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Multiplicity-one representations of divergence-free Lie algebras

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ABSTRACT

Divergence-free Lie algebras (also known as the special Lie algebras of Cartan type) are Lie algebras of volume-preserving transformation groups. They are simple in generic case. Dokovic and Zhao found a certain graded generalization of them. In this paper, we classify all the irreducible and indecomposable multiplicity-one modules of the simple generalized divergence-free Lie algebras.

1. Introduction

Lie algebras of Cartan type are closely related to geometry and dynamics. They also play important roles in the structure theory of simple Lie algebras. Kac [5] gave an abstract definition of generalized Cartan type Lie algebras by derivations. After that, various graded generalizations of Lie algebras of Cartan type were constructed and studied by Kawamoto [8], Osborn [11], Dokovic and Zhao [1–3], and Zhao [27]. The fundamental ingredients for these graded generalizations are the pairs of a semi-group algebra and a set of grading operators and down-grading operators. Other nongraded Cartan type Lie algebras were studied in [24–26,12,13].

The representations of simple Lie algebras of Cartan type have attracted the attention of many researchers as well. Shen [16–18] introduced mixed product of graded modules over graded Lie algebras of Cartan type and obtained certain irreducible modules over a field with characteristic p. Larsson [9] obtained the same representation of Witt algebras from different motivation (later was called "Larsson

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functor" by some authors). Rao [14,15] studied irreducibility of the weight modules over the derivation Lie algebra of the algebra of Laurent polynomials virtually constructed by Shen in [16] (Larsson functor).

Lin and Tan [10] constructed some irreducible weight modules over the Lie algebra of quantum torus via Shen's mixed product. Zhao [29] determined the module structure of Shen's mixed product over Xu's nongraded Lie algebras of Witt type (cf. [25]). Moreover, she [30] constructed a family of irreducible modules for Xu's two-derivation nongraded Lie algebras of Block type (cf. [24]) by restricting Shen's module to these algebras. Furthermore, she [31] obtained a composition series for a family of modules with parameters over Xu's nongraded Hamiltonian Lie algebras (cf. [25]). These modules are constructed from finite-dimensional multiplicity-one irreducible modules of symplectic Lie algebras by Shen's mixed product. In [32], Zhao and Liang generalized the results of [30] to Xu's four-derivation nongraded Lie algebras of Block type (cf. [24]).

Howe [4] classified all the finite-dimensional multiplicity-one representations over finite-dimensional simple Lie algebras. On infinite-dimensional simple Lie algebra side, Kaplansky [6,7] and Santharoubane [7] gave classifications of multiplicity-one representations over classical Virasoro algebras. After that, Su [19,20] generalized Kaplansky and Santharoubane's result to multiplicity-one modules over high rank Virasoro algebras and super-Virasoro algebras. Based on their classifications of multiplicity-one representations over generalized Virasoro algebras in [23], Zhao [28] classified the multiplicity-one representations over graded generalized Witt algebras. Moreover, Su and Zhou [22] generalized Zhao's result in [28] to generalized weight modules over the nongraded generalized Witt algebras introduced by Xu [25].

Divergence-free Lie algebras, which are also called the *special Lie algebras of Cartan type*, are Lie algebras of volume-preserving transformation groups. They are simple in generic case. In this paper, we classify all the irreducible and indecomposable multiplicity-one graded modules over the simple generalized divergence-free Lie algebras introduced by Dokovic and Zhao [3]. Since the algebras we are concerned with do not contain a centerless Virasoro algebra, we cannot use the existing results and have to find a new way to do our classification. This is a reason why this paper is so long. Below are some details.

Let Γ be an additive group. A Lie algebra $\mathcal{G}$ over a field $\mathbb{F}$ is called a *Γ-graded Lie algebra* if $\mathcal{G}=\bigoplus_{\alpha\in\Gamma}\mathcal{G}_\alpha$, where each $\mathcal{G}_\alpha$ is a subspace of $\mathcal{G}$, and $[\mathcal{G}_\alpha,\mathcal{G}_\beta]\subseteq\mathcal{G}_{\alpha+\beta}$. A module M over a Γ-graded Lie algebra $\mathcal{G}$ is called a *graded module* if $M=\bigoplus_{\alpha\in\Gamma}M_\alpha$ is a Γ-graded space and $\mathcal{G}_\alpha M_\beta\subset M_{\alpha+\beta}$. Each component M_α is called a *homogenous subspace* of M. The aim of this paper is to give a complete classification of the irreducible and indecomposable graded modules of the simple generalized divergence-free Lie algebras introduced by Dokovic and Zhao [3], whose dimensions of homogenous subspaces are less than or equal to one. In a subsequent paper, we will classify all the irreducible and indecomposable multiplicity-one generalized graded modules over Xu's nongraded divergence-free Lie algebras (cf. [25,21]) as Su and Zhou did for Xu's nongraded Witt algebras in [22].

Throughout this paper, we assume that the base field $\mathbb{F}$ is an algebraically closed field with characteristic 0. This paper is organized as follows. In Section 2, we review the definitions and some basic facts about the generalized Witt algebras given by Kawamoto [8] and the generalized divergence-free Lie algebras introduced by Dokovic and Zhao [1,2]. Moreover, we construct three classes of graded modules over Dokovic and Zhao's generalized divergence-free Lie algebras with 1-dimensional homogenous subspaces, and give the necessary and sufficient conditions for two such modules to be isomorphic. Furthermore, we give the main theorem of classifying all the irreducible and indecomposable graded modules of these divergence-free Lie algebras, whose dimensions of homogenous subspaces are less than or equal to one. In Sections 3 to 5, we prove the main theorem under different conditions, respectively.

2. The Lie algebras and modules

We first review the definition of generalized Witt algebras introduced by Kawamoto [8]. The definition here is different from that in [8], but they are equivalent.

Let D be a finite-dimensional nonzero vector space and let Γ be an additive subgroup of its dual space D^* such that $\bigcap_{\alpha\in\Gamma}\ker\alpha=\{0\}$. Denote by $\mathbb{F}[\Gamma]$ the vector space with a basis $\{x^\alpha\mid\alpha\in\Gamma\}$ and the multiplication determined by $x^\alpha x^\beta = x^{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. Write 1 instead of x^0 for convenience. The tensor product $\mathcal{W} = \mathbb{F}[\Gamma] \otimes_{\mathbb{F}} D$ is a free left $\mathbb{F}[\Gamma]$-module. Moreover, we denote an arbitrary element of D by ∂, and for simplicity, we write $x^\alpha \partial$ instead of $x^\alpha \otimes \partial$. Under the bracket

$$\left[x^\alpha \partial_1, x^\beta \partial_2\right] = x^{\alpha+\beta}\left(\beta(\partial_1)\partial_2 - \alpha(\partial_2)\partial_1\right) \quad \text{for any } \partial_1, \partial_2 \in D \text{ and } \alpha, \beta \in \Gamma, \tag{2.1}$$

$\mathcal{W}$ becomes a Lie algebra. It is usually denoted by $\mathcal{W}(\Gamma, D)$, and is called a *generalized Witt algebra*. Kawamoto [8] showed that $\mathcal{W}(\Gamma, D)$ is a simple Lie algebra.

The Lie algebra $\mathcal{W} = \mathcal{W}(\Gamma, D)$ has a natural Γ-gradation $\mathcal{W} = \bigoplus_{\alpha\in\Gamma} \mathcal{W}_\alpha$, where $\mathcal{W}_\alpha = x^\alpha D$ for $\alpha \in \Gamma$. In particular, we have $\mathcal{W}_0 = D$. It follows that

$$\left[\partial, x^\alpha \partial_1\right] = \alpha(\partial) x^\alpha \partial_1, \quad \forall \partial, \partial_1 \in D,\ \alpha \in \Gamma, \tag{2.2}$$

namely, $\operatorname{ad}\partial$ acts on $\mathcal{W}_\alpha$ as the scalar $\alpha(\partial)$. Hence D is a toral Cartan subalgebra of $\mathcal{W}$.

We now review the generalized divergence-free Lie algebras introduced by Dokovic and Zhao [3]. Define the divergence as the $\mathbb{F}$-linear map div from $\mathcal{W}$ to $\mathbb{F}[\Gamma]$ determined by

$$\operatorname{div}\left(x^\alpha \partial\right) = \alpha(\partial) x^\alpha, \quad \forall \alpha \in \Gamma,\ \partial \in D. \tag{2.3}$$

Note that $\mathbb{F}[\Gamma]$ becomes a $\mathcal{W}$-module with the action:

$$x^\alpha \partial . x^\beta = \beta(\partial) x^{\alpha+\beta} \quad \text{for } \partial \in D,\ \alpha, \beta \in \Gamma. \tag{2.4}$$

It is easy to verify that the divergence has the following two properties:

$$\operatorname{div}[u, v] = u.\operatorname{div}(v) - v.\operatorname{div}(u), \tag{2.5}$$

$$\operatorname{div}(f w) = f \operatorname{div}(w) + w.f \tag{2.6}$$

for $u, v, w \in \mathcal{W}$ and $f \in \mathbb{F}[\Gamma]$. So the subspace

$$\tilde{\mathcal{S}} = \ker(\operatorname{div}) \tag{2.7}$$

forms a subalgebra of $\mathcal{W}$. If $\dim D = 1$, then $\tilde{\mathcal{S}} = D$. We assume from now on that $\dim D \geqslant 2$. Then we have that

$$\tilde{\mathcal{S}} = \bigoplus_{\alpha\in\Gamma} \tilde{\mathcal{S}}_\alpha, \quad \text{where } \tilde{\mathcal{S}}_\alpha = \tilde{\mathcal{S}} \cap \mathcal{W}_\alpha. \tag{2.8}$$

It is clear that $\tilde{\mathcal{S}}_\alpha = x^\alpha(\ker\alpha)$. Consequently

$$\operatorname{codim}_{\mathcal{W}_\alpha}(\tilde{\mathcal{S}}_\alpha) = 1 - \delta_{\alpha,0}. \tag{2.9}$$

We set $\mathcal{S} = [\tilde{\mathcal{S}}, \tilde{\mathcal{S}}]$. Then

$$\mathcal{S} = \bigoplus_{\alpha\in\Gamma\setminus\{0\}} \tilde{\mathcal{S}}_\alpha \tag{2.10}$$

and $\mathcal{S}$ is a simple Lie algebra (cf. [3]). The Lie algebra $\mathcal{S}$ is called a *simple generalized divergence-free Lie algebra* (also called a *generalized Cartan type special Lie algebra*). The goal of this paper is to classify all the irreducible and indecomposable multiplicity-one graded modules of the Lie algebra $\mathcal{S}$. We sometimes denote $\mathcal{S}$ by $\mathcal{S}(\Gamma, D)$.

For any $\mu, \eta \in D^*$, we define the Γ-graded $\mathcal{S}$-module $\mathscr{M}_\mu = \bigoplus_{\alpha\in\Gamma} \mathbb{F}v_\alpha$ with the action

$$x^\alpha \partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \quad \text{for any } \alpha \in \Gamma\backslash\{0\},\ \beta \in \Gamma,\ \partial \in \ker\alpha; \tag{2.11}$$

the Γ-graded $\mathcal{S}$-module $\mathscr{A}_\eta = \bigoplus_{\alpha\in\Gamma} \mathbb{F}v_\alpha$ with the action

$$\begin{aligned} x^\alpha \partial.v_\beta &= \beta(\partial)v_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Gamma\backslash\{0\},\ \partial \in \ker\alpha, \\ x^\alpha \partial.v_0 &= \eta(\partial)v_\alpha \quad \text{for } \alpha \in \Gamma\backslash\{0\},\ \partial \in \ker\alpha; \end{aligned} \tag{2.12}$$

the Γ-graded $\mathcal{S}$-module $\mathscr{B}_\eta = \bigoplus_{\alpha\in\Gamma} \mathbb{F}v_\alpha$ with the action

$$\begin{aligned} x^\alpha \partial.v_\beta &= \beta(\partial)v_{\alpha+\beta} \quad \text{for } \alpha \in \Gamma\backslash\{0\},\ \beta \in \Gamma\backslash\{-\alpha\},\ \partial \in \ker\alpha, \\ x^\alpha \partial.v_{-\alpha} &= \eta(\partial)v_0 \quad \text{for } \alpha \in \Gamma\backslash\{0\},\ \partial \in \ker\alpha. \end{aligned} \tag{2.13}$$

It is easy to verify that these three classes of modules are restrictions to $\mathcal{S}$ of Zhao's five classes of modules over $\mathcal{W}$ in [28]. It is straightforward to prove:

Lemma 2.1. *The irreducibility of the three classes of modules $\mathscr{M}_\mu$, $\mathscr{A}_\eta$ and $\mathscr{B}_\eta$ is as follows*:

(i) *$\mathscr{M}_\mu$ is irreducible if $\mu \notin \Gamma$. When $\mu \in \Gamma$, $\mathscr{M}_\mu = (\bigoplus_{\alpha\in\Gamma\backslash\{-\mu\}} \mathbb{F}v_\alpha) \oplus \mathbb{F}v_{-\mu}$ is a direct sum of two irreducible submodules.*
(ii) *If $\eta \neq 0$, the modules $\mathscr{A}_\eta$ and $\mathscr{B}_\eta$ are indecomposable but reducible. In this case, $\mathscr{A}_\eta$ has the irreducible submodule $\bigoplus_{\alpha\in\Gamma\backslash\{0\}} \mathbb{F}v_\alpha$, while $\mathscr{B}_\eta$ has the 1-dimensional trivial submodule $\mathbb{F}v_0$ and the quotient $\mathscr{B}_\eta/\mathbb{F}v_0$ is irreducible. If $\eta = 0$, then $\mathscr{A}_0 \simeq \mathscr{B}_0 \simeq \mathscr{M}_0$.*

We use $\mathscr{M}'_\mu$, $\mathscr{A}'_\eta$ and $\mathscr{B}'_\eta$ to denote the nontrivial irreducible submodules or the nontrivial irreducible quotients of $\mathscr{M}_\mu$, $\mathscr{A}_\eta$ and $\mathscr{B}_\eta$, respectively. In analogy with Zhao's result, we have:

Theorem 2.2. *Among the $\mathcal{S}$-modules $\mathscr{M}_\mu$, $\mathscr{A}_\eta$, $\mathscr{B}_\eta$ for $\mu, \eta \in D^*$, and their nontrivial irreducible submodules or nontrivial irreducible quotients, we have only the following module isomorphisms*:

(i) *$\mathscr{M}_\mu \simeq \mathscr{M}_{\mu'}$ iff $\mu - \mu' \in \Gamma$,*
(ii) *$\mathscr{M}'_\mu \simeq \mathscr{M}'_{\mu'}$ iff $\mu - \mu' \in \Gamma$,*
(iii) *$\mathscr{A}_\eta \simeq \mathscr{A}_{a\eta}$ for any $a \in \mathbb{F}\backslash\{0\}$,*
(iv) *$\mathscr{B}_\eta \simeq \mathscr{B}_{a\eta}$ for any $a \in \mathbb{F}\backslash\{0\}$,*
(v) *$\mathscr{A}_0 \simeq \mathscr{B}_0 \simeq \mathscr{M}_0$,*
(vi) *$\mathscr{A}'_\eta \simeq \mathscr{B}'_\eta \simeq \mathscr{M}'_0$.*

Then we give our main result of this paper:

Theorem 2.3. *Suppose* $\dim D \geqslant 3$. *Assume that $V = \bigoplus_{\theta\in\Gamma} V_\theta$ is a Γ-graded $\mathcal{S}(\Gamma, D)$-module with* $\dim V_\theta \leqslant 1$ *for $\theta \in \Gamma$. If V is irreducible or indecomposable, then V is isomorphic to one of the following modules for appropriate $\mu \in D^*$ and $\eta \in D^*\backslash\{0\}$*:

(i) *the trivial module* $\mathbb{F}v_0$; (ii) $\mathscr{M}'_\mu$; (iii) $\mathscr{A}_\eta$; (iv) $\mathscr{B}_\eta$.

When $\dim D = 2$, $\mathcal{S}(\Gamma, D)$ is simultaneously a graded Hamiltonian Lie algebra. We guess that Theorem 2.3 also holds for the case $\dim D = 2$. However, when $\dim D = 2$, there are less acting relations of its corresponding algebras on their modules. Our method in this paper doesn't apply to the case

$\dim D = 2$. We need another way of solving it and doing the classification of modules of graded Hamiltonian Lie algebras.

In the following sections, we will prove Theorem 2.3 case by case progressively.

3. The case $\dim D = 3$ and $\Gamma \simeq \mathbb{Z}^3$

In this section, we will prove Theorem 2.3 under the condition that $\dim D = 3$ and $\Gamma \simeq \mathbb{Z}^3$. Throughout this section, we shall always assume that $\dim D = 3$ and $\Gamma \simeq \mathbb{Z}^3$.

Let $M = \bigoplus_{\theta \in \Gamma} M_\theta$ be a Γ-graded $\mathcal{S}$-module with $\dim M_\theta = 1$ for each $\theta \in \Gamma$. Write $M_\theta = \mathbb{F} w_\theta$ for $\theta \in \Gamma$. In order to prove Theorem 2.3, we first specify all the possible action of $\mathcal{S}(\Gamma, D)$ on M. We start our analysis with two useful properties.

Fix any nonzero $\sigma \in \Gamma$. Take any two nonzero vectors $\partial, \partial' \in \ker \sigma$. It can be noticed that $x^\sigma \partial$ and $x^{-\sigma} \partial'$ preserve $\bigoplus_{i \in \mathbb{Z}} M_{\nu + i\sigma}$ for any $\nu \in \Gamma$. Fix ν and we write

$$x^{-\sigma} \partial' . x^\sigma \partial . w_{\nu + i\sigma} = c_i w_{\nu + i\sigma} \quad \text{with } c_i \in \mathbb{F} \text{ for } i \in \mathbb{Z}. \tag{3.1}$$

We demonstrate how $x^\sigma \partial$ and $x^{-\sigma} \partial'$ act on $\bigoplus_{i \in \mathbb{Z}} M_{\nu + i\sigma}$ in the following lemma.

Lemma 3.1. *The constant $c_i = c$ is independent of i. Moreover, if $c \neq 0$, for $a \in \{\sqrt{c}, -\sqrt{c}\}$, there exist $\{0 \neq v_{\nu + i\sigma} \in M_{\nu + i\sigma} \mid i \in \mathbb{Z}\}$ such that $x^\sigma \partial . v_{\nu + (i-1)\sigma} = a v_{\nu + i\sigma}$ and $x^{-\sigma} \partial' . v_{\nu + i\sigma} = a v_{\nu + (i-1)\sigma}$ for $i \in \mathbb{Z}$.*

Proof. Note $x^\sigma \partial$ and $x^{-\sigma} \partial'$ commute. For any $i \in \mathbb{Z}$, we have

$$\begin{aligned} c_i x^\sigma \partial . w_{\nu + i\sigma} &= x^\sigma \partial . \big(x^{-\sigma} \partial' . x^\sigma \partial . w_{\nu + i\sigma}\big) = x^{-\sigma} \partial' . x^\sigma \partial . \big(x^\sigma \partial . w_{\nu + i\sigma}\big) \\ &= c_{i+1} x^\sigma \partial . w_{\nu + i\sigma}. \end{aligned} \tag{3.2}$$

Thus $c_i = c_{i+1}$ if $x^\sigma \partial . w_{\nu + i\sigma} \neq 0$. When $x^\sigma \partial . w_{\nu + i\sigma} = 0$, we obtain

$$x^{-\sigma} \partial' . x^\sigma \partial . w_{\nu + i\sigma} = 0 \quad \text{and} \quad x^{-\sigma} \partial' . x^\sigma \partial . w_{\nu + (i+1)\sigma} = x^\sigma \partial . \big(x^{-\sigma} \partial' . w_{\nu + (i+1)\sigma}\big) = 0, \tag{3.3}$$

which indicates $c_i = 0 = c_{i+1}$. To sum up, we get $c_i = c_{i+1}$ for any $i \in \mathbb{Z}$, i.e., $c_i = c$ is independent of i.

Suppose $c \neq 0$. The fact $c_i = c$ implies that $x^\sigma \partial . w_{\nu + i\sigma} \neq 0$ for all $i \in \mathbb{Z}$. Take $v_\nu = w_\nu$. Fixing $a \in \{\sqrt{c}, -\sqrt{c}\}$, we define $v_{\nu + i\sigma}$'s by $x^\sigma \partial . v_{\nu + (i-1)\sigma} = a v_{\nu + i\sigma}$ for $i \in \mathbb{Z}$. Then

$$x^{-\sigma} \partial' . v_{\nu + i\sigma} = \frac{1}{a} x^{-\sigma} \partial' . \big(x^\sigma \partial . v_{\nu + (i-1)\sigma}\big) = a v_{\nu + (i-1)\sigma}. \tag{3.4}$$

So the lemma follows. □

Let $\sigma, \rho \in \Gamma$ be any $\mathbb{Z}$-linearly independent elements. Observe that $\dim(\ker \sigma \cap \ker \rho) = 1$. Pick any nonzero vector $\tilde{\partial}_1 \in \ker \sigma \cap \ker \rho$. Take $\tilde{\partial}_2 \in \ker \sigma \setminus \mathbb{F}\tilde{\partial}_1$ and $\tilde{\partial}_3 \in \ker \rho \setminus \mathbb{F}\tilde{\partial}_1$. Then $\{\tilde{\partial}_1, \tilde{\partial}_2\}$ forms a basis of $\ker \sigma$ and $\{\tilde{\partial}_1, \tilde{\partial}_3\}$ forms a basis of $\ker \rho$. Obviously $\rho(\tilde{\partial}_2) \neq 0$ and $\sigma(\tilde{\partial}_3) \neq 0$. Notice that $x^{\pm\sigma} \tilde{\partial}_1$, $x^{\pm\sigma} \tilde{\partial}_2$, $x^{\pm\rho} \tilde{\partial}_1$ and $x^{\pm\rho} \tilde{\partial}_3$ preserve $\bigoplus_{i,k \in \mathbb{Z}} M_{\nu + i\sigma + k\rho}$ for any $\nu \in \Gamma$. We derive their action on $\bigoplus_{i,k \in \mathbb{Z}} M_{\nu + i\sigma + k\rho}$ under certain conditions in the following lemma.

Lemma 3.2. *If $x^{-\sigma} \tilde{\partial}_1 . x^\sigma \tilde{\partial}_1 . w_\nu \neq 0$ and $x^{-\rho} \tilde{\partial}_1 . x^\rho \tilde{\partial}_1 . w_\nu \neq 0$ for some $\nu \in \Gamma$, then there exist $\{0 \neq v_{\nu + i\sigma + k\rho} \in M_{\nu + i\sigma + k\rho} \mid i, k \in \mathbb{Z}\}$ such that*

$$x^{\pm\sigma} \tilde{\partial}_1 . v_{\nu + i\sigma + k\rho} = a_1 v_{\nu + (i \pm 1)\sigma + k\rho}, \qquad x^{\pm\rho} \tilde{\partial}_1 . v_{\nu + i\sigma + k\rho} = b_1 v_{\nu + i\sigma + (k \pm 1)\rho} \tag{3.5}$$

for $i, k \in \mathbb{Z}$, where a_1 and b_1 are any nonzero constants satisfying $x^{-\sigma}\tilde{\partial}_1.x^{\sigma}\tilde{\partial}_1.w_\nu = a_1^2 w_\nu$ and $x^{-\rho}\tilde{\partial}_1.x^{\rho}\tilde{\partial}_1.w_\nu = b_1^2 w_\nu$, respectively. For such $v_{\nu+i\sigma+k\rho}$'s, writing $x^{\sigma}\tilde{\partial}_2.v_\nu = a_2 v_{\nu+\sigma}$, $x^{\rho}\tilde{\partial}_3.v_\nu = a_3 v_{\nu+\rho}$ and $x^{\sigma+\rho}\tilde{\partial}_1.v_\nu = d v_{\nu+\sigma+\rho}$, we have the following relations:

$$x^{\sigma+\rho}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho} = d v_{\nu+(i+1)\sigma+(k+1)\rho} \quad \text{for } i, k \in \mathbb{Z}; \tag{3.6}$$

$$x^{\pm\sigma}\tilde{\partial}_2.v_{\nu+i\sigma+k\rho} = \left(a_2 + k\rho(\tilde{\partial}_2)\frac{d}{b_1}\right) v_{\nu+(i\pm1)\sigma+k\rho} \quad \text{for } i, k \in \mathbb{Z}; \tag{3.7}$$

$$x^{\pm\rho}\tilde{\partial}_3.v_{\nu+i\sigma+k\rho} = \left(a_3 + i\sigma(\tilde{\partial}_3)\frac{d}{a_1}\right) v_{\nu+i\sigma+(k\pm1)\rho} \quad \text{for } i, k \in \mathbb{Z}. \tag{3.8}$$

Moreover, we have $a_1^2 = b_1^2 = d^2$.

Proof. Since $x^{-\sigma}\tilde{\partial}_1.x^{\sigma}\tilde{\partial}_1.w_\nu \neq 0$ and $x^{-\rho}\tilde{\partial}_1.x^{\rho}\tilde{\partial}_1.w_\nu \neq 0$, Lemma 3.1 enables us to choose $\{0 \neq v_{\nu+i\sigma} \in M_{\nu+i\sigma} \mid i \in \mathbb{Z}\}$ and $\{0 \neq v_{\nu+k\rho} \in M_{\nu+k\rho} \mid k \in \mathbb{Z}\}$ such that

$$x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma} = a_1 v_{\nu+(i\pm1)\sigma} \quad \text{and} \quad x^{\pm\rho}\tilde{\partial}_1.v_{\nu+k\rho} = b_1 v_{\nu+(k\pm1)\rho} \tag{3.9}$$

for $i, k \in \mathbb{Z}$, where a_1 and b_1 are any nonzero constants satisfying $x^{-\sigma}\tilde{\partial}_1.x^{\sigma}\tilde{\partial}_1.w_\nu = a_1^2 w_\nu$ and $x^{-\rho}\tilde{\partial}_1.x^{\rho}\tilde{\partial}_1.w_\nu = b_1^2 w_\nu$, respectively.

In order to determine the other v's, we shall first prove $x^{\rho}\tilde{\partial}_1.w_{\nu+i\sigma+k\rho} \neq 0$ for all $i, k \in \mathbb{Z}$. Consider the case $i, k \geqslant 0$. We give the proof by induction on i. If $i = 0$, (3.9) shows

$$x^{\rho}\tilde{\partial}_1.w_{\nu+k\rho} \neq 0 \quad \text{for all } k \geqslant 0. \tag{3.10}$$

For some $i \geqslant 0$, we assume

$$x^{\rho}\tilde{\partial}_1.w_{\nu+i\sigma+k\rho} \neq 0 \quad \text{for all } k \geqslant 0. \tag{3.11}$$

Since $x^{-\sigma}\tilde{\partial}_1$ and $x^{\rho}\tilde{\partial}_1$ commute, (3.9) and (3.11) indicate

$$x^{-\sigma}\tilde{\partial}_1.x^{\rho}\tilde{\partial}_1.v_{\nu+(i+1)\sigma} = x^{\rho}\tilde{\partial}_1.\left(x^{-\sigma}\tilde{\partial}_1.v_{\nu+(i+1)\sigma}\right) = a_1 x^{\rho}\tilde{\partial}_1.v_{\nu+i\sigma} \neq 0, \tag{3.12}$$

which implies $x^{\rho}\tilde{\partial}_1.v_{\nu+(i+1)\sigma} \neq 0$ and $x^{-\sigma}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+\rho} \neq 0$. Suppose $x^{\rho}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+k\rho} \neq 0$ and $x^{-\sigma}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+1)\rho} \neq 0$ for some $k \geqslant 0$. Then this together with (3.11) indicates

$$x^{-\sigma}\tilde{\partial}_1.x^{\rho}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+1)\rho} = x^{\rho}\tilde{\partial}_1.x^{-\sigma}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+1)\rho} \neq 0, \tag{3.13}$$

which implies $x^{\rho}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+1)\rho} \neq 0$ and $x^{-\sigma}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+2)\rho} \neq 0$. By induction on k, we have

$$x^{\rho}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+k\rho} \neq 0 \quad \text{and} \quad x^{-\sigma}\tilde{\partial}_1.w_{\nu+(i+1)\sigma+(k+1)\rho} \neq 0 \quad \text{for all } k \geqslant 0. \tag{3.14}$$

So induction on i gives

$$x^{\rho}\tilde{\partial}_1.w_{\nu+i\sigma+k\rho} \neq 0 \quad \text{for } i, k \geqslant 0. \tag{3.15}$$

It can be proved similarly for the other cases: $i, k \leqslant 0$; $i \geqslant 0$, $k \leqslant 0$; $i \leqslant 0$, $k \geqslant 0$. We omit the details. So we have

$$x^{\rho}\tilde{\partial}_1.w_{\nu+i\sigma+k\rho} \neq 0 \quad \text{for } i, k \in \mathbb{Z}. \tag{3.16}$$

Now we are ready to define the v's. Take $v_{\nu+i\sigma}$ and $v_{\nu+k\rho}$ for $i, k \in \mathbb{Z}$ as they were in (3.9). Define $v_{\nu+i\sigma+k\rho}$'s with $i, k \in \mathbb{Z}\backslash\{0\}$ by

$$x^{\rho}\tilde{\partial}_1.v_{\nu+i'\sigma+(k'-1)\rho} = b_1 v_{\nu+i'\sigma+k'\rho} \quad \text{for } i', k' \in \mathbb{Z} \text{ with } i' \neq 0. \tag{3.17}$$

Then it follows from (3.9) and (3.17) that

$$x^{\rho}\tilde{\partial}_1.v_{\nu+i\sigma+(k-1)\rho} = b_1 v_{\nu+i\sigma+k\rho} \quad \text{for } i, k \in \mathbb{Z}. \tag{3.18}$$

For $k > 0$, (3.18) implies

$$\begin{aligned} x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho} &= x^{\pm\sigma}\tilde{\partial}_1.\left(\frac{x^{\rho}\tilde{\partial}_1}{b_1}\right)^k.v_{\nu+i\sigma} = \left(\frac{x^{\rho}\tilde{\partial}_1}{b_1}\right)^k.x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma} \\ &= a_1 v_{\nu+(i\pm1)\sigma+k\rho}. \end{aligned} \tag{3.19}$$

Moreover,

$$\begin{aligned} \left(\frac{x^{\rho}\tilde{\partial}_1}{b_1}\right)^k.\left(x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma-k\rho}\right) &= x^{\pm\sigma}\tilde{\partial}_1.\left(\frac{x^{\rho}\tilde{\partial}_1}{b_1}\right)^k.v_{\nu+i\sigma-k\rho} \\ &= x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma} = a_1 v_{\nu+(i\pm1)\sigma} \end{aligned} \tag{3.20}$$

for $k > 0$, which together with (3.18) indicates $x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma-k\rho} = a_1 v_{\nu+(i\pm1)\sigma-k\rho}$. So we have

$$x^{\pm\sigma}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho} = a_1 v_{\nu+(i\pm1)\sigma+k\rho} \quad \text{for } i, k \in \mathbb{Z}. \tag{3.21}$$

Similarly, it can be deduced from (3.9) and (3.21) that

$$x^{-\rho}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho} = b_1 v_{\nu+i\sigma+(k-1)\rho} \quad \text{for } i, k \in \mathbb{Z}. \tag{3.22}$$

Finally we want to show that $x^{\sigma+\rho}\tilde{\partial}_1$, $x^{\pm\sigma}\tilde{\partial}_2$ and $x^{\pm\rho}\tilde{\partial}_3$ act on the v's in the desired expressions.

Note $[x^{\sigma+\rho}\tilde{\partial}_1, x^{\pm\sigma}\tilde{\partial}_1] = 0$ and $[x^{\sigma+\rho}\tilde{\partial}_1, x^{\pm\rho}\tilde{\partial}_1] = 0$. From the assumption $x^{\sigma+\rho}\tilde{\partial}_1.v_\nu = dv_{\nu+\sigma+\rho}$, it can be derived that

$$x^{\sigma+\rho}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho} = dv_{\nu+(i+1)\sigma+(k+1)\rho} \quad \text{for } i, k \in \mathbb{Z}. \tag{3.23}$$

Moreover, since

$$\left[x^{-\sigma}\tilde{\partial}_2, x^{\sigma+\rho}\tilde{\partial}_1\right].v_\nu = \rho(\tilde{\partial}_2)x^{\rho}\tilde{\partial}_1.v_\nu = b_1\rho(\tilde{\partial}_2)v_{\nu+\rho} \neq 0, \tag{3.24}$$

we have $d \neq 0$.

Write

$$x^{\sigma}\tilde{\partial}_2.v_{\nu+i\sigma+k\rho} = f^{+}(i,k)v_{\nu+(i+1)\sigma+k\rho} \quad \text{for } i,k\in\mathbb{Z}. \tag{3.25}$$

Using (3.21) in $[x^{\sigma}\tilde{\partial}_2, x^{\sigma}\tilde{\partial}_1].v_{\nu+i\sigma+k\rho} = 0$, we get $f^{+}(i+1,k) = f^{+}(i,k)$. Moreover, applying (3.17) and (3.23) to

$$\bigl[x^{\sigma}\tilde{\partial}_2, x^{\rho}\tilde{\partial}_1\bigr].v_{\nu+i\sigma+k\rho} = \rho(\tilde{\partial}_2)x^{\sigma+\rho}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho}, \tag{3.26}$$

we derive $f^{+}(i,k+1) - f^{+}(i,k) = \rho(\tilde{\partial}_2)\frac{d}{b_1}$. Therefore, by induction, we get

$$f^{+}(i,k) = f^{+}(0,0) + k\rho(\tilde{\partial}_2)\frac{d}{b_1} \quad \text{for } i,k\in\mathbb{Z}. \tag{3.27}$$

Write

$$x^{-\sigma}\tilde{\partial}_2.v_{\nu+i\sigma+k\rho} = f^{-}(i,k)v_{\nu+(i-1)\sigma+k\rho} \quad \text{for } i,k\in\mathbb{Z}. \tag{3.28}$$

Then using (3.21) in $[x^{-\sigma}\tilde{\partial}_2, x^{\sigma}\tilde{\partial}_1].v_{\nu+i\sigma+k\rho} = 0$, we get $f^{-}(i+1,k) = f^{-}(i,k)$. Moreover, substituting (3.17) and (3.23) into

$$\bigl[x^{-\sigma}\tilde{\partial}_2, x^{\sigma+\rho}\tilde{\partial}_1\bigr].v_{\nu+i\sigma+k\rho} = \rho(\tilde{\partial}_2)x^{\rho}\tilde{\partial}_1.v_{\nu+i\sigma+k\rho}, \tag{3.29}$$

we see that $f^{-}(i+1,k+1) - f^{-}(i,k) = \rho(\tilde{\partial}_2)\frac{b_1}{d}$. Therefore, by induction, we obtain

$$f^{-}(i,k) = f^{-}(0,0) + k\rho(\tilde{\partial}_2)\frac{b_1}{d} \quad \text{for } i,k\in\mathbb{Z}. \tag{3.30}$$

Set

$$x^{\rho}\tilde{\partial}_3.v_{\nu+i\sigma+k\rho} = g^{+}(i,k)v_{\nu+i\sigma+(k+1)\rho}$$

and

$$x^{-\rho}\tilde{\partial}_3.v_{\nu+i\sigma+k\rho} = g^{-}(i,k)v_{\nu+i\sigma+(k-1)\rho} \tag{3.31}$$

for $i,k\in\mathbb{Z}$. Observe that $[x^{\rho}\tilde{\partial}_3, x^{\rho}\tilde{\partial}_1] = 0$ and $[x^{\rho}\tilde{\partial}_3, x^{\sigma}\tilde{\partial}_1] = \sigma(\tilde{\partial}_3)x^{\sigma+\rho}\tilde{\partial}_1$. As (3.27), we deduce

$$g^{+}(i,k) = g^{+}(0,0) + i\sigma(\tilde{\partial}_3)\frac{d}{a_1} \quad \text{for } i,k\in\mathbb{Z} \tag{3.32}$$

by induction. Likewise, $[x^{-\rho}\tilde{\partial}_3, x^{\rho}\tilde{\partial}_1] = 0$, $[x^{-\rho}\tilde{\partial}_3, x^{\sigma+\rho}\tilde{\partial}_1] = \sigma(\tilde{\partial}_3)x^{\sigma}\tilde{\partial}_1$ and induction give rise to

$$g^{-}(i,k) = g^{-}(0,0) + i\sigma(\tilde{\partial}_3)\frac{a_1}{d} \quad \text{for } i,k\in\mathbb{Z}. \tag{3.33}$$

We get $b_1^2 = d^2$ by applying (3.17), (3.21), (3.30) and (3.33) to

$$\bigl[\bigl[x^{-\sigma}\tilde{\partial}_2, x^{\rho}\tilde{\partial}_1\bigr], x^{-\rho}\tilde{\partial}_3\bigr].v_{\nu} = \rho(\tilde{\partial}_2)\sigma(\tilde{\partial}_3)x^{-\sigma}\tilde{\partial}_1.v_{\nu}. \tag{3.34}$$

Then substituting (3.27), (3.30), (3.32) and $b_1^2 = d^2$ into

$$\big[x^{-\sigma}\tilde{\partial}_2, \big[x^{\sigma}\tilde{\partial}_2, x^{\rho}\tilde{\partial}_3\big]\big].v_\nu = \rho^2(\tilde{\partial}_2)x^{\rho}\tilde{\partial}_3.v_\nu, \tag{3.35}$$

we get $f^-(0,0) = f^+(0,0)$. Moreover, (3.27), (3.30) and $b_1^2 = d^2$ show

$$f^-(i,k) = f^+(i,k) = a_2 + k\rho(\tilde{\partial}_2)\frac{d}{b_1} \quad \text{for } i,k \in \mathbb{Z}. \tag{3.36}$$

We obtain $a_1^2 = d^2$ by inserting (3.21), (3.22), (3.27) and (3.32) into

$$\big[x^{\rho}\tilde{\partial}_3, \big[x^{-\rho}\tilde{\partial}_1, x^{\sigma}\tilde{\partial}_2\big]\big].v_\nu = \rho(\tilde{\partial}_2)\sigma(\tilde{\partial}_3)x^{\sigma}\tilde{\partial}_1.v_\nu. \tag{3.37}$$

Applying (3.32), (3.33), (3.36) and $a_1^2 = d^2$ to

$$\big[x^{-\rho}\tilde{\partial}_3, \big[x^{\rho}\tilde{\partial}_3, x^{\sigma}\tilde{\partial}_2\big]\big].v_\nu = \sigma^2(\tilde{\partial}_3)x^{\sigma}\tilde{\partial}_2.v_\nu, \tag{3.38}$$

we deduce $g^-(0,0) = g^+(0,0)$. Moreover, (3.32), (3.33) and $a_1^2 = d^2$ indicate

$$g^-(i,k) = g^+(i,k) = a_3 + i\sigma(\tilde{\partial}_3)\frac{d}{a_1} \quad \text{for } i,k \in \mathbb{Z}. \tag{3.39}$$

Thus the lemma follows. □

Next we begin to analyze the possible action of $\mathcal{S}$ on M.

Fix a $\mathbb{Z}$-basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of Γ. Then $\ker\varepsilon_1 \cap \ker\varepsilon_2 \cap \ker\varepsilon_3 = \{0\}$. Picking any nonzero vectors $\partial_1 \in \ker\varepsilon_2 \cap \ker\varepsilon_3$, $\partial_2 \in \ker\varepsilon_1 \cap \ker\varepsilon_3$ and $\partial_3 \in \ker\varepsilon_1 \cap \ker\varepsilon_2$, we get an $\mathbb{F}$-basis of D. Obviously $\varepsilon_1(\partial_1) \neq 0$, $\varepsilon_2(\partial_2) \neq 0$ and $\varepsilon_3(\partial_3) \neq 0$. It can be verified that

$$\mathcal{S}(\Gamma, D) \text{ is generated by } \mathcal{X} = \big\{x^{\pm\varepsilon_1}\partial_2, x^{\pm\varepsilon_1}\partial_3, x^{\pm\varepsilon_2}\partial_1, x^{\pm\varepsilon_2}\partial_3, x^{\pm\varepsilon_3}\partial_1, x^{\pm\varepsilon_3}\partial_2\big\}. \tag{3.40}$$

Thus, we only need to derive how $\mathcal{X}$ acts on M. We analyze the action of $\mathcal{X}$ case by case in the following lemmas.

Lemma 3.3. *If there exists some $\nu \in \Gamma$ such that*

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\nu \neq 0 \quad \textit{and} \quad x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\nu \neq 0,$$

or,

$$x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_\nu \neq 0 \quad \textit{and} \quad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_\nu \neq 0,$$

or,

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_\nu \neq 0 \quad \textit{and} \quad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_\nu \neq 0,$$

then M is isomorphic to one of the following modules for appropriate $\mu \in D^*$ and $\eta \in D^*\backslash\{0\}$:

(i) $\mathscr{M}_\mu$; (ii) $\mathscr{A}_\eta$; (iii) $\mathscr{B}_\eta$.

Proof. Assume $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\nu \neq 0$ and $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\nu \neq 0$ for some $\nu \in \Gamma$; the other cases can be proved similarly. By a translation of the indices, we may assume $\nu = 0$. Our two main goals in the proof are to properly choose

$$\{0 \neq v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \in M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \mid i, j, k \in \mathbb{Z}\} \tag{3.41}$$

and to derive how $\mathcal{X}$ acts on them (cf. (3.40)). The main idea is that we first specify the action of $\mathcal{X}$ on the three subspaces $\bigoplus_{j,k\in\mathbb{Z}} M_{j\varepsilon_2+k\varepsilon_3}$, $\bigoplus_{i,k\in\mathbb{Z}} M_{i\varepsilon_1+k\varepsilon_3}$ and $\bigoplus_{i,j\in\mathbb{Z}} M_{i\varepsilon_1+j\varepsilon_2}$, and then extend it to the action of $\mathcal{X}$ on M. We process our proof in several steps.

Step 1. Elements $\{v_{j\varepsilon_2+k\varepsilon_3} \mid j, k \in \mathbb{Z}\}$ and action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_3}\partial_1$, $x^{\pm\varepsilon_3}\partial_2$.

Since $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0 \neq 0$ and $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0 \neq 0$, Lemma 3.2 enables us to choose $\{0 \neq v_{j\varepsilon_2+k\varepsilon_3} \in M_{j\varepsilon_2+k\varepsilon_3} \mid j, k \in \mathbb{Z}\}$ such that

$$x^{\pm\varepsilon_2}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} = a_1 v_{(j\pm1)\varepsilon_2+k\varepsilon_3} \quad \text{and} \quad x^{\pm\varepsilon_3}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} = b_1 v_{j\varepsilon_2+(k\pm1)\varepsilon_3}, \tag{3.42}$$

where a_1 and b_1 are nonzero scalars satisfying $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.v_0 = a_1^2 v_0$ and $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.v_0 = b_1^2 v_0$, respectively. Write $x^{\varepsilon_2+\varepsilon_3}\partial_1.v_0 = d_1 v_{\varepsilon_2+\varepsilon_3}$. Lemma 3.2 says $a_1^2 = b_1^2 = d_1^2$. Changing the sign of $v_{j\varepsilon_2+k\varepsilon_3}$ for $j, k \in \mathbb{Z}$ if necessary, we can take $a_1 = b_1 = d_1$; equivalently, we get $\{0 \neq v_{j\varepsilon_2+k\varepsilon_3} \in M_{j\varepsilon_2+k\varepsilon_3} \mid j, k \in \mathbb{Z}\}$ such that

$$x^{\pm\varepsilon_2}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} = a_1 v_{(j\pm1)\varepsilon_2+k\varepsilon_3}, \tag{3.43}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} = a_1 v_{j\varepsilon_2+(k\pm1)\varepsilon_3}, \tag{3.44}$$

$$x^{\varepsilon_2+\varepsilon_3}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} = a_1 v_{(j+1)\varepsilon_2+(k+1)\varepsilon_3}, \tag{3.45}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big) v_{j\varepsilon_2+(k\pm1)\varepsilon_3}, \tag{3.46}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big) v_{(j\pm1)\varepsilon_2+k\varepsilon_3}, \tag{3.47}$$

where a_2 and a_3 are constants determined by $x^{\varepsilon_3}\partial_2.v_0 = a_2 v_{\varepsilon_3}$ and $x^{\varepsilon_2}\partial_3.v_0 = a_3 v_{\varepsilon_2}$.

Step 2. Elements $\{v_{i\varepsilon_1+k\varepsilon_3}, v_{i\varepsilon_1+j\varepsilon_2} \mid i, j, k \in \mathbb{Z}\}$ and action of the set $\mathcal{X}$ (cf. (3.40)).

According to (3.43) and (3.44), $x^{\pm\varepsilon_2}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} \neq 0$ and $x^{\pm\varepsilon_3}\partial_1.v_{j\varepsilon_2+k\varepsilon_3} \neq 0$ for any $j, k \in \mathbb{Z}$. By another proper translation of the indices, we may assume that

$$a_2 + j\varepsilon_2(\partial_2) \neq 0 \quad \text{for } |j| \leqslant 1 \quad \text{and} \quad a_3 + k\varepsilon_3(\partial_3) \neq 0 \quad \text{for } |k| \leqslant 1. \tag{3.48}$$

Then we consider applying Lemma 3.2 to the action of $\mathcal{X}$ on the subspaces $\bigoplus_{i,k\in\mathbb{Z}} M_{i\varepsilon_1+k\varepsilon_3}$ and $\bigoplus_{i,j\in\mathbb{Z}} M_{i\varepsilon_1+j\varepsilon_2}$. First it can be deduced from (3.48) that

$$x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.v_0 = a_2^2 v_0 \neq 0 \quad \text{and} \quad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.v_0 = a_3^2 v_0 \neq 0. \tag{3.49}$$

Next we need the following claim.

Claim 1. *$x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.v_0 \neq 0$ and $x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.v_0 \neq 0$.*

We only give the proof of $x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.v_0 \neq 0$. Inequality $x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.v_0 \neq 0$ can be similarly proved.

Firstly, we want to prove $x^{\varepsilon_1}\partial_2.v_0 \neq 0$. Assume $x^{\varepsilon_1}\partial_2.v_0 = 0$ and then we will see this leads to a contradiction. Since $a_2 \neq 0$, we have

$$x^{\varepsilon_1}\partial_2.v_{\varepsilon_3} = \frac{1}{a_2}x^{\varepsilon_1}\partial_2.x^{\varepsilon_3}\partial_2.v_0 = \frac{1}{a_2}x^{\varepsilon_3}\partial_2.x^{\varepsilon_1}\partial_2.v_0 = 0, \tag{3.50}$$

which further implies

$$x^{\varepsilon_1+\varepsilon_3}\partial_2.v_0 = \frac{1}{\varepsilon_1(\partial_1)}\big[x^{\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_2\big].v_0 = -\frac{a_1}{\varepsilon_1(\partial_1)}x^{\varepsilon_1}\partial_2.v_{\varepsilon_3} = 0. \tag{3.51}$$

If $x^{-\varepsilon_1}\partial_3.v_0 = 0$, it can be derived

$$0 \neq a_2 v_{\varepsilon_3} = x^{\varepsilon_3}\partial_2.v_0 = \frac{1}{\varepsilon_3(\partial_3)}\big[x^{-\varepsilon_1}\partial_3, x^{\varepsilon_1+\varepsilon_3}\partial_2\big].v_0 = 0, \tag{3.52}$$

which is absurd. Consider $x^{-\varepsilon_1}\partial_3.v_0 \neq 0$. Since

$$x^{\varepsilon_1}\partial_2.\big(x^{-\varepsilon_1}\partial_3.v_0\big) = x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_2.v_0 = 0, \tag{3.53}$$

we obtain $x^{\varepsilon_1}\partial_2.w_{-\varepsilon_1} = 0$, which further implies

$$x^{-\varepsilon_3}\partial_2.\big(x^{\varepsilon_1}\partial_2.w_{-\varepsilon_1+\varepsilon_3}\big) = x^{\varepsilon_1}\partial_2.\big(x^{-\varepsilon_3}\partial_2.w_{-\varepsilon_1+\varepsilon_3}\big) = 0. \tag{3.54}$$

As $x^{-\varepsilon_3}\partial_2.v_{\varepsilon_3} = a_2 v_0 \neq 0$, (3.54) leads to $x^{\varepsilon_1}\partial_2.w_{-\varepsilon_1+\varepsilon_3} = 0$. Then

$$x^{\varepsilon_1+\varepsilon_3}\partial_2.w_{-\varepsilon_1} = \frac{1}{\varepsilon_1(\partial_1)}\big[x^{\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_2\big].w_{-\varepsilon_1} = -\frac{1}{\varepsilon_1(\partial_1)}x^{\varepsilon_1}\partial_2.\big(x^{\varepsilon_3}\partial_1.w_{-\varepsilon_1}\big) = 0. \tag{3.55}$$

From (3.51) and (3.55), it can be deduced

$$0 \neq a_2 v_{\varepsilon_3} = x^{\varepsilon_3}\partial_2.v_0 = \frac{1}{\varepsilon_3(\partial_3)}\big[x^{-\varepsilon_1}\partial_3, x^{\varepsilon_1+\varepsilon_3}\partial_2\big].v_0 = 0, \tag{3.56}$$

which leads to a contradiction. Hence we have

$$x^{\varepsilon_1}\partial_2.v_0 \neq 0. \tag{3.57}$$

Secondly, we want to prove $x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1} \neq 0$. Suppose $x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1} = 0$. Then we will get a contradiction. Note

$$x^{-\varepsilon_3}\partial_2.\big(x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1+\varepsilon_3}\big) = x^{-\varepsilon_1}\partial_2.\big(x^{-\varepsilon_3}\partial_2.w_{\varepsilon_1+\varepsilon_3}\big) = 0 \tag{3.58}$$

and $x^{-\varepsilon_3}\partial_2.v_{\varepsilon_3} = a_2 v_0 \neq 0$. We therefore get $x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1+\varepsilon_3} = 0$. If $x^{-\varepsilon_1}\partial_2.v_0 = 0$, we derive

$$\begin{aligned} 0 \neq a_2 v_{\varepsilon_3} = x^{\varepsilon_3}\partial_2.v_0 &= \frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{\varepsilon_1+\varepsilon_3}\big(\varepsilon_1(\partial_1)\partial_3 - \varepsilon_3(\partial_3)\partial_1\big), x^{-\varepsilon_1}\partial_2\big].v_0 \\ &= 0, \end{aligned} \tag{3.59}$$

which is a contradiction. On the other hand, we assume $x^{-\varepsilon_1}\partial_2.v_0 \neq 0$. Since $x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1} = 0$ leads to

$$x^{\varepsilon_1}\partial_3.\big(x^{-\varepsilon_1}\partial_2.v_0\big) = x^{-\varepsilon_1}\partial_2.\big(x^{\varepsilon_1}\partial_3.v_0\big) = 0, \tag{3.60}$$

we get $x^{\varepsilon_1}\partial_3.w_{-\varepsilon_1} = 0$. Since

$$x^{-\varepsilon_3}\partial_2.x^{-\varepsilon_1}\partial_2.v_{\varepsilon_3} = x^{-\varepsilon_1}\partial_2.x^{-\varepsilon_3}\partial_2.v_{\varepsilon_3} = a_2x^{-\varepsilon_1}\partial_2.v_0 \neq 0, \tag{3.61}$$

we obtain $x^{-\varepsilon_1}\partial_2.v_{\varepsilon_3} \neq 0$. By $x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1+\varepsilon_3} = 0$ mentioned above, we get

$$x^{\varepsilon_1}\partial_3.\big(x^{-\varepsilon_1}\partial_2.v_{\varepsilon_3}\big) = x^{-\varepsilon_1}\partial_2.\big(x^{\varepsilon_1}\partial_3.v_{\varepsilon_3}\big) = 0, \tag{3.62}$$

which implies $x^{\varepsilon_1}\partial_3.w_{-\varepsilon_1+\varepsilon_3} = 0$. Then we have

$$\begin{aligned} 0 \neq \varepsilon_3(\partial_3)\varepsilon_1(\partial_1)x^{\varepsilon_3}\partial_2.v_0 &= \big[\big[x^{\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_3\big], x^{-\varepsilon_1}\partial_2\big].v_0 \\ &= x^{\varepsilon_3}\partial_1.x^{\varepsilon_1}\partial_3.\big(x^{-\varepsilon_1}\partial_2.v_0\big) - x^{\varepsilon_1}\partial_3.\big(x^{\varepsilon_3}\partial_1.x^{-\varepsilon_1}\partial_2.v_0\big) \\ &\quad - x^{-\varepsilon_1}\partial_2.\big(\big[x^{\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_3\big].v_0\big) = 0, \end{aligned} \tag{3.63}$$

which is a contradiction. So we must have

$$x^{-\varepsilon_1}\partial_2.w_{\varepsilon_1} \neq 0. \tag{3.64}$$

In summary, (3.57) and (3.64) give $x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.v_0 \neq 0$. Symmetrically, considering the action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_1}\partial_2$ and $x^{\pm\varepsilon_1}\partial_3$ on the subspace $\bigoplus_{i,j\in\mathbb{Z}} M_{i\varepsilon_1+j\varepsilon_2}$, we can similarly prove $x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.v_0 \neq 0$. Hence Claim 1 holds.

Now we can apply Lemma 3.2 to the action of $\mathcal{X}$ on the subspaces $\bigoplus_{i,k\in\mathbb{Z}} M_{i\varepsilon_1+k\varepsilon_3}$ and $\bigoplus_{i,j\in\mathbb{Z}} M_{i\varepsilon_1+j\varepsilon_2}$.

Observe that Claim 1 and (3.49) give $x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.v_0 \neq 0$ and $x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.v_0 \neq 0$. Take $\{v_{k\varepsilon_3} \mid k \in \mathbb{Z}\}$ as they were in Step 1. Then Lemma 3.2 enables us to choose $\{0 \neq v_{i\varepsilon_1+k\varepsilon_3} \in M_{i\varepsilon_1+k\varepsilon_3} \mid i,k \in \mathbb{Z},\ i \neq 0\}$ such that

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+k\varepsilon_3} = a_2v_{(i\pm1)\varepsilon_1+k\varepsilon_3}, \qquad x^{\pm\varepsilon_3}\partial_2.v_{i\varepsilon_1+k\varepsilon_3} = a_2v_{i\varepsilon_1+(k\pm1)\varepsilon_3} \tag{3.65}$$

for $i,k \in \mathbb{Z}$. Write $x^{\varepsilon_1+\varepsilon_3}\partial_2.v_0 = d_2v_{\varepsilon_1+\varepsilon_3}$ and $x^{\varepsilon_1}\partial_3.v_0 = bv_{\varepsilon_1}$. Lemma 3.2 gives that

$$x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{i\varepsilon_1+k\varepsilon_3} = d_2v_{(i+1)\varepsilon_1+(k+1)\varepsilon_3}, \tag{3.66}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{i\varepsilon_1+k\varepsilon_3} = \left(a_1 + i\varepsilon_1(\partial_1)\frac{d_2}{a_2}\right)v_{i\varepsilon_1+(k\pm1)\varepsilon_3} \tag{3.67}$$

and

$$x^{\pm\varepsilon_1}\partial_3.v_{i\varepsilon_1+k\varepsilon_3} = \left(b + k\varepsilon_3(\partial_3)\frac{d_2}{a_2}\right)v_{(i\pm1)\varepsilon_1+k\varepsilon_3} \tag{3.68}$$

for $i,k \in \mathbb{Z}$. Moreover, we have $a_2^2 = d_2^2$.

Note that Claim 1 and (3.49) show $x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.v_0 \neq 0$ and $x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.v_0 \neq 0$. Take $\{v_{j\varepsilon_2} \mid j \in \mathbb{Z}\}$ and $\{v_{i\varepsilon_1} \mid i \in \mathbb{Z}\backslash\{0\}\}$ as they were in Step 1 and (3.65), respectively. Then Lemma 3.2 says that there exist $\{0 \neq v_{i\varepsilon_1+j\varepsilon_2} \in M_{i\varepsilon_1+j\varepsilon_2} \mid i,j \in \mathbb{Z}\backslash\{0\}\}$ such that

$$x^{\pm\varepsilon_1}\partial_3.v_{i\varepsilon_1+j\varepsilon_2}=bv_{(i\pm1)\varepsilon_1+j\varepsilon_2},\qquad x^{\pm\varepsilon_2}\partial_3.v_{i\varepsilon_1+j\varepsilon_2}=a_3v_{i\varepsilon_1+(j\pm1)\varepsilon_2} \tag{3.69}$$

for $i,j\in\mathbb{Z}$. Writing $x^{\varepsilon_1+\varepsilon_2}\partial_3.v_0=d_3v_{\varepsilon_1+\varepsilon_2}$, we have

$$x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{i\varepsilon_1+j\varepsilon_2}=d_3v_{(i+1)\varepsilon_1+(j+1)\varepsilon_2}, \tag{3.70}$$

$$x^{\pm\varepsilon_2}\partial_1.v_{i\varepsilon_1+j\varepsilon_2}=\left(a_1+i\varepsilon_1(\partial_1)\frac{d_3}{b}\right)v_{i\varepsilon_1+(j\pm1)\varepsilon_2}, \tag{3.71}$$

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2}=\left(a_2+j\varepsilon_2(\partial_2)\frac{d_3}{a_3}\right)v_{(i\pm1)\varepsilon_1+j\varepsilon_2} \tag{3.72}$$

for $i,j\in\mathbb{Z}$ again by Lemma 3.2. Moreover, $a_3^2=b^2=d_3^2$.

Among the constants a_2, d_2, a_3, b and d_3, we have the following relations.

Claim 2. *$a_2=d_2$ and $a_3=b=d_3$.*

Let i be a nonzero integer such that $a_1+i\varepsilon_1(\partial_1)\frac{d_2}{a_2}\neq0$ and $a_1+i\varepsilon_1(\partial_1)\frac{d_3}{b}\neq0$. From (3.67) and (3.71) we deduce

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.v_{i\varepsilon_1}=\left(a_1+i\varepsilon_1(\partial_1)\frac{d_2}{a_2}\right)^2v_{i\varepsilon_1}\neq0 \tag{3.73}$$

and

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.v_{i\varepsilon_1}=\left(a_1+i\varepsilon_1(\partial_1)\frac{d_3}{b}\right)^2v_{i\varepsilon_1}\neq0. \tag{3.74}$$

Then Lemma 3.2 implies $(a_1+i\varepsilon_1(\partial_1)\frac{d_2}{a_2})^2=(a_1+i\varepsilon_1(\partial_1)\frac{d_3}{b})^2$, which gives $\frac{d_2}{a_2}=\frac{d_3}{b}$. Similarly, by (3.46), (3.72) and Lemma 3.2, we find $(a_2+j\varepsilon_2(\partial_2))^2=(a_2+j\varepsilon_2(\partial_2)\frac{d_3}{a_3})^2$ for some $j\neq0$, which implies $a_3=d_3$.

Moreover, $a_2^2=d_2^2$ indicates $\frac{d_2}{a_2}=\pm1$. To prove the claim, it suffices to show $\frac{d_2}{a_2}=1$. Suppose $\frac{d_2}{a_2}=-1$, and then we will see this leads to a contradiction.

Since $a_1+\varepsilon_1(\partial_1)\neq0$ or $a_1-\varepsilon_1(\partial_1)\neq0$, we may assume $a_1-\varepsilon_1(\partial_1)\neq0$; the other case can be proved similarly. By (3.67), (3.71) and the assumption $\frac{d_2}{a_2}=\frac{d_3}{b}=-1$, we have

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.v_{\varepsilon_1}=\big(a_1-\varepsilon_1(\partial_1)\big)^2v_{\varepsilon_1}\neq0, \tag{3.75}$$

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.v_{\varepsilon_1}=\big(a_1-\varepsilon_1(\partial_1)\big)^2v_{\varepsilon_1}\neq0. \tag{3.76}$$

Take $\{v_{\varepsilon_1+k\varepsilon_3}\mid k\in\mathbb{Z}\}$ and $\{v_{\varepsilon_1+j\varepsilon_2}\mid j\in\mathbb{Z}\}$ as they were in (3.65) and (3.69), respectively. According to Lemma 3.2, we can choose $\{0\neq v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\in M_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\mid j,k\in\mathbb{Z}\backslash\{0\}\}$ such that

$$x^{\pm\varepsilon_2}\partial_1.v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1-\varepsilon_1(\partial_1)\big)v_{\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3}, \tag{3.77}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1-\varepsilon_1(\partial_1)\big)v_{\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3} \tag{3.78}$$

for $j,k\in\mathbb{Z}$. Moreover, Lemma 3.2, (3.65) and (3.69) give

$$x^{\varepsilon_2+\varepsilon_3}\partial_1.v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=s\big(a_1-\varepsilon_1(\partial_1)\big)v_{\varepsilon_1+(j+1)\varepsilon_2+(k+1)\varepsilon_3},\tag{3.79}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+sj\varepsilon_2(\partial_2)\big)v_{\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3},\tag{3.80}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+sk\varepsilon_3(\partial_3)\big)v_{\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3}\tag{3.81}$$

for $j,k\in\mathbb{Z}$, where $s\in\{1,-1\}$ is to be specified. Substituting (3.46), (3.72), (3.80) and $a_3=d_3$ into $[x^{\varepsilon_1}\partial_2,x^{\varepsilon_3}\partial_2].v_{\varepsilon_2}=0$, and making use of (3.48), we get

$$x^{\varepsilon_1}\partial_2.v_{\varepsilon_2+\varepsilon_3}=\big(a_2+s\varepsilon_2(\partial_2)\big)v_{\varepsilon_1+\varepsilon_2+\varepsilon_3}.\tag{3.82}$$

Using (3.46), (3.72), (3.80), (3.82) and $a_3=d_3$ in $[x^{\varepsilon_1}\partial_2,x^{-\varepsilon_3}\partial_2].v_{\varepsilon_2+\varepsilon_3}=0$, we find

$$\big(a_2+\varepsilon_2(\partial_2)\big)^2=\big(a_2+s\varepsilon_2(\partial_2)\big)^2,\tag{3.83}$$

which implies $s=1$. Then (3.44), (3.72), (3.78), (3.82) and $a_3=d_3$ give

$$x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{\varepsilon_2}=\frac{1}{\varepsilon_1(\partial_1)}\big[x^{\varepsilon_3}\partial_1,x^{\varepsilon_1}\partial_2\big].v_{\varepsilon_2}=-\big(a_2+\varepsilon_2(\partial_2)\big)v_{\varepsilon_1+\varepsilon_2+\varepsilon_3}.\tag{3.84}$$

By $[x^{\varepsilon_1+\varepsilon_3}\partial_2,x^{\varepsilon_2}\partial_1].v_0=[x^{\varepsilon_1}\partial_2,x^{\varepsilon_2+\varepsilon_3}\partial_1].v_0$, we get

$$a_1\varepsilon_2(\partial_2)+a_2\varepsilon_1(\partial_1)=0.\tag{3.85}$$

Applying it to $[x^{-\varepsilon_1}\partial_2,[x^{\varepsilon_1}\partial_2,x^{\varepsilon_2}\partial_1]].v_0=\varepsilon_2^2(\partial_2)x^{\varepsilon_2}\partial_1.v_0$, we obtain

$$a_2\varepsilon_1(\partial_1)\varepsilon_2(\partial_2)=0,\tag{3.86}$$

which is absurd. So we must have $\frac{d_2}{a_2}=\frac{d_3}{b}=1$, from which it follows

$$a_2=d_2,\qquad a_3=b=d_3.\tag{3.87}$$

Therefore Claim 2 holds.

Step 3. Extension to $\{v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\mid i,j,k\in\mathbb{Z}\}$ and general action of $\mathcal{X}$.

In Step 1 and Step 2 we have properly chosen $\{v_{j\varepsilon_2+k\varepsilon_3},v_{i\varepsilon_1+k\varepsilon_3},v_{i\varepsilon_1+j\varepsilon_2}\mid i,j,k\in\mathbb{Z}\}$ and determined part of the action of $\mathcal{X}$ (see (3.43)–(3.47), (3.65)–(3.72) and Claim 2). Now we start to extend it to the general action of $\mathcal{X}$.

As (3.67), (3.71) and Claim 2 show

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.v_{i\varepsilon_1}=\big(a_1+i\varepsilon_1(\partial_1)\big)^2v_{i\varepsilon_1}\tag{3.88}$$

and

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.v_{i\varepsilon_1}=\big(a_1+i\varepsilon_1(\partial_1)\big)^2v_{i\varepsilon_1},\tag{3.89}$$

we can apply Lemma 3.2 to the action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_3}\partial_1$ and $x^{\pm\varepsilon_3}\partial_2$ on the subspaces

$$\bigoplus_{j,k\in\mathbb{Z}}M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\quad\text{for } i\in\mathbb{Z}\backslash\{0\}\text{ such that } a_1+i\varepsilon_1(\partial_1)\neq0.\tag{3.90}$$

Seeing that there exists at most one $i\in\mathbb{Z}$ such that $a_1+i\varepsilon_1(\partial_1)=0$, we denote it by i'. Obviously $i'\neq 0$. Fix any $i\in\mathbb{Z}\backslash\{0,i'\}$. Then Lemma 3.2 enables us to choose $\{0\neq v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\in M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\mid j,k\in\mathbb{Z}\backslash\{0\}\}$ such that

$$x^{\pm\varepsilon_2}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1+i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3},\tag{3.91}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1+i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3}\tag{3.92}$$

for $j,k\in\mathbb{Z}$. Lemma 3.2 further implies

$$x^{\varepsilon_2+\varepsilon_3}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=s_i\big(a_1+i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+(j+1)\varepsilon_2+(k+1)\varepsilon_3},\tag{3.93}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+s_ij\varepsilon_2(\partial_2)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3},\tag{3.94}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+s_ik\varepsilon_3(\partial_3)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3}\tag{3.95}$$

for $j,k\in\mathbb{Z}$, where $s_i\in\{1,-1\}$ is to be determined.

Set $s_0=1$. Then (3.43)–(3.47) coincide with (3.91)–(3.95).

If i' does not exist, (3.91)–(3.95) together with (3.43)–(3.47) show the general action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_3}\partial_1$ and $x^{\pm\varepsilon_3}\partial_2$. To complete this step, we only need to determine s_i for $i\in\mathbb{Z}\backslash\{0\}$ and to specify the general action of $x^{\pm\varepsilon_1}\partial_2$ and $x^{\pm\varepsilon_1}\partial_3$.

If i' does exist, except for determining s_i for $i\in\mathbb{Z}\backslash\{0,i'\}$, we need to determine $\{v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\mid j,k\in\mathbb{Z}\backslash\{0\}\}$ and to specify the action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_3}\partial_1$ and $x^{\pm\varepsilon_3}\partial_2$ on the subspace $\bigoplus_{j,k\in\mathbb{Z}}M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$. Moreover, the general action of $x^{\pm\varepsilon_1}\partial_2$ and $x^{\pm\varepsilon_1}\partial_3$ needs to be determined.

We give the details in the following.

Case 1. i' does not exist.

To begin with, we want to determine s_i for $i\in\mathbb{Z}\backslash\{0\}$. Let p be any fixed integer. Applying (3.72) and (3.94) to $[x^{\varepsilon_1}\partial_2,x^{\varepsilon_3}\partial_2].v_{p\varepsilon_1+\varepsilon_2}=0$, we have

$$x^{\varepsilon_1}\partial_2.v_{p\varepsilon_1+\varepsilon_2+\varepsilon_3}=\frac{(a_2+\varepsilon_2(\partial_2))(a_2+s_{p+1}\varepsilon_2(\partial_2))}{a_2+s_p\varepsilon_2(\partial_2)}v_{(p+1)\varepsilon_1+\varepsilon_2+\varepsilon_3}.\tag{3.96}$$

Moreover, substituting (3.72), (3.94) and (3.96) into $[x^{\varepsilon_1}\partial_2,x^{-\varepsilon_3}\partial_2].v_{p\varepsilon_1+\varepsilon_2+\varepsilon_3}=0$, and making use of (3.48), we obtain

$$\big(a_2+s_p\varepsilon_2(\partial_2)\big)^2=\big(a_2+s_{p+1}\varepsilon_2(\partial_2)\big)^2,\tag{3.97}$$

which implies $s_p=s_{p+1}$. Since $p\in\mathbb{Z}$ is arbitrary and $s_0=1$, induction shows

$$s_i=1\quad\text{for all } i\in\mathbb{Z}.\tag{3.98}$$

So (3.91)–(3.95) together with (3.43)–(3.47) show

$$x^{\pm\varepsilon_2}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1+i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3},\tag{3.99}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_1+i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3},\tag{3.100}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3},\tag{3.101}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3}\tag{3.102}$$

for $i,j,k\in\mathbb{Z}$.

Next we want to derive the general action of $x^{\pm\varepsilon_1}\partial_2$ and $x^{\pm\varepsilon_1}\partial_3$.

We first consider determining the action of $x^{\pm\varepsilon_1}\partial_2$. Recall that (3.72) and Claim 2 give

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{(i\pm1)\varepsilon_1+j\varepsilon_2}\quad\text{for } i,j\in\mathbb{Z}. \tag{3.103}$$

Since

$$x^{\pm\varepsilon_1}\partial_2.x^{\pm\varepsilon_3}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=x^{\pm\varepsilon_3}\partial_2.x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.104}$$

Eq. (3.101) and induction on k lead to

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{(i\pm1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \tag{3.105}$$

for $i,k\in\mathbb{Z}$ and $j\in\mathbb{Z}$ such that $a_2+j\varepsilon_2(\partial_2)\neq0$. As there exists at most one $j\in\mathbb{Z}$ such that $a_2+j\varepsilon_2(\partial_2)=0$, we denote it by j'. If j' exists, then (3.101) implies

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=\pm\frac{1}{\varepsilon_1(\partial_1)\varepsilon_3(\partial_3)}\big[x^{-\varepsilon_3}\partial_1,\big[x^{\pm\varepsilon_1}\partial_3,x^{\varepsilon_3}\partial_2\big]\big].v_{i\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=0 \tag{3.106}$$

for $i,k\in\mathbb{Z}$, which coincides with (3.105). So no matter such j' exists or not, we have

$$x^{\pm\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{(i\pm1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\quad\text{for } i,j,k\in\mathbb{Z}. \tag{3.107}$$

It holds similarly that

$$x^{\pm\varepsilon_1}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{(i\pm1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\quad\text{for } i,j,k\in\mathbb{Z}. \tag{3.108}$$

To sum up, (3.99)–(3.102) together with (3.107) and (3.108) show the action of $\mathcal{X}$ on M. Define $\mu\in D^*$ by $\mu(\partial_1)=a_1$, $\mu(\partial_2)=a_2$ and $\mu(\partial_3)=a_3$. Then we can write the action of $\mathcal{X}$ uniformly by

$$x^{\epsilon}\partial.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=(\mu+i\varepsilon_1+j\varepsilon_2+k\varepsilon_3)(\partial)v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3+\epsilon} \tag{3.109}$$

for $i,j,k\in\mathbb{Z}$, $\epsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\}$ and $\partial\in\ker\epsilon\cap\{\partial_1,\partial_2,\partial_3\}$. Since $\mathcal{X}$ generates $\mathcal{S}(\Gamma,D)$ (cf. (3.40)), from (3.109) we deduce

$$M\simeq\mathscr{M}_{\mu}. \tag{3.110}$$

Case 2. i' does exist.

Recall that j' denotes the possible integer satisfying $a_2+j'\varepsilon_2(\partial_2)=0$. Likewise, we denote by k' the possible integer satisfying $a_3+k'\varepsilon_3(\partial_3)=0$.

Firstly, we want to determine s_i for $i\in\mathbb{Z}\backslash\{0,i'\}$ and obtain:

(1) the action of $x^{\pm\varepsilon_2}\partial_1$, $x^{\pm\varepsilon_2}\partial_3$, $x^{\pm\varepsilon_3}\partial_1$ and $x^{\pm\varepsilon_3}\partial_2$ on $\bigoplus_{i,j,k\in\mathbb{Z};\,i\neq i'}M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(2) the action of $x^{\varepsilon_1}\partial_2$ and $x^{\varepsilon_1}\partial_3$ on $\bigoplus_{i,j,k\in\mathbb{Z};\,i\neq i',i'-1}M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(3) the action of $x^{-\varepsilon_1}\partial_2$ and $x^{-\varepsilon_1}\partial_3$ on $\bigoplus_{i,j,k\in\mathbb{Z};\,i\neq i',i'+1}M_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$.

Since (3.96) and (3.97) also hold for $p \in \mathbb{Z}\backslash\{i', i'-1\}$ in this case, we have $s_p = s_{p+1}$ for $p \in \mathbb{Z}\backslash\{i', i'-1\}$. Moreover, we shall show $s_{i'-1} = s_{i'+1}$. As (3.71), (3.72) and Claim 2 indicate

$$\begin{aligned} x^{2\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2} &= \frac{1}{4\varepsilon_1^2(\partial_1)\varepsilon_2(\partial_2)}\big[\big[x^{\varepsilon_1}\partial_2, \big[x^{\varepsilon_1}\partial_2, x^{\varepsilon_2}\partial_1\big]\big], x^{-\varepsilon_2}\partial_1\big].v_{i\varepsilon_1+j\varepsilon_2} \\ &= \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{(i+2)\varepsilon_1+j\varepsilon_2} \quad \text{for } i, j \in \mathbb{Z}, \end{aligned} \tag{3.111}$$

applying it and (3.94) to $[x^{2\varepsilon_1}\partial_2, x^{\varepsilon_3}\partial_2].v_{(i'-1)\varepsilon_1+\varepsilon_2} = 0$, we get

$$x^{2\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+\varepsilon_2+\varepsilon_3} = \frac{(a_2+\varepsilon_2(\partial_2))(a_2+s_{i'+1}\varepsilon_2(\partial_2))}{a_2+s_{i'-1}\varepsilon_2(\partial_2)}v_{(i'+1)\varepsilon_1+\varepsilon_2+\varepsilon_3}. \tag{3.112}$$

Then inserting (3.94), (3.111) and (3.112) into $[x^{2\varepsilon_1}\partial_2, x^{-\varepsilon_3}\partial_2].v_{(i'-1)\varepsilon_1+\varepsilon_2+\varepsilon_3} = 0$, and making use of (3.48), we obtain

$$\big(a_2 + s_{i'-1}\varepsilon_2(\partial_2)\big)^2 = \big(a_2 + s_{i'+1}\varepsilon_2(\partial_2)\big)^2, \tag{3.113}$$

which implies $s_{i'-1} = s_{i'+1}$. Thus this and $s_p = s_{p+1}$ for $p \in \mathbb{Z}\backslash\{i', i'-1\}$ indicate

$$s_i = s_0 = 1 \quad \text{for } i \in \mathbb{Z}\backslash\big\{i'\big\}. \tag{3.114}$$

So (3.91)–(3.95) together with (3.43)–(3.47) show

$$x^{\pm\varepsilon_2}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_1 + i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3}, \tag{3.115}$$

$$x^{\pm\varepsilon_3}\partial_1.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_1 + i\varepsilon_1(\partial_1)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3}, \tag{3.116}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{i\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3}, \tag{3.117}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{i\varepsilon_1+(j\pm1)\varepsilon_2+k\varepsilon_3} \tag{3.118}$$

for $i, j, k \in \mathbb{Z}$ with $i \neq i'$.

Similar arguments as those from (3.103) to (3.108) show

$$x^{\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{(i+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } i, j, k \in \mathbb{Z},\ i \neq i', i'-1, \tag{3.119}$$

$$x^{\varepsilon_1}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{(i+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } i, j, k \in \mathbb{Z},\ i \neq i', i'-1, \tag{3.120}$$

$$x^{-\varepsilon_1}\partial_2.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{(i-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } i, j, k \in \mathbb{Z},\ i \neq i', i'+1, \tag{3.121}$$

$$x^{-\varepsilon_1}\partial_3.v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{(i-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } i, j, k \in \mathbb{Z},\ i \neq i', i'+1. \tag{3.122}$$

Secondly, we want to define $\{v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \mid j \in \mathbb{Z}\backslash\{0, j'\},\ k \in \mathbb{Z}\backslash\{0\}\}$, *and derive*

(1) the action of $x^{\pm\varepsilon_3}\partial_1$ and $x^{\pm\varepsilon_3}\partial_2$ on $\bigoplus_{j,k\in\mathbb{Z};\ j\neq j'} M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(2) the action of $x^{\varepsilon_1}\partial_2$ and $x^{\varepsilon_1}\partial_3$ on $\bigoplus_{j,k\in\mathbb{Z};\ j\neq j'} M_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(3) the action of $x^{-\varepsilon_1}\partial_2$ and $x^{-\varepsilon_1}\partial_3$ on $\bigoplus_{j,k\in\mathbb{Z};\ j\neq j'} M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(4) the action of $x^{\varepsilon_2}\partial_1$ and $x^{\varepsilon_2}\partial_3$ on $\bigoplus_{j,k\in\mathbb{Z};\ j\neq j', j'-1} M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(5) the action of $x^{-\varepsilon_2}\partial_1$ and $x^{-\varepsilon_2}\partial_3$ on $\bigoplus_{j,k\in\mathbb{Z};\ j\neq j', j'+1} M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$.

To define the v's, we first need to prove $x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\neq 0$ for $j,k\in\mathbb{Z}$ with $j\neq j'$. Fix any $j\in\mathbb{Z}\backslash\{j'\}$. From (3.72) and Claim 2 we know that

$$x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2}\neq 0. \tag{3.123}$$

Since (3.117) leads to

$$\begin{aligned}x^{\mp\varepsilon_3}\partial_2.x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k\pm 1)\varepsilon_3}&=x^{\varepsilon_1}\partial_2.\big(x^{\mp\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k\pm 1)\varepsilon_3}\big)\\&=\big(a_2+j\varepsilon_2(\partial_2)\big)x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3},\end{aligned}\tag{3.124}$$

we have $x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\neq 0$ for $k\in\mathbb{Z}$ by (3.123) and induction on k. So we get

$$x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\neq 0\quad\text{for } j,k\in\mathbb{Z},\ j\neq j'. \tag{3.125}$$

Now we can define $\{v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\mid j\in\mathbb{Z}\backslash\{0,j'\},\ k\in\mathbb{Z}\backslash\{0\}\}$ by

$$x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}. \tag{3.126}$$

This together with (3.65) and (3.72) indicate

$$x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\quad\text{for } j,k\in\mathbb{Z},\ j\neq j'. \tag{3.127}$$

Next we shall derive the other action in (1)–(5).

From applying (3.117) and (3.127) to

$$x^{\pm\varepsilon_3}\partial_2.x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=x^{\varepsilon_1}\partial_2.x^{\pm\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.128}$$

it can be immediately deduced

$$x^{\pm\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2+(k\pm 1)\varepsilon_3}\quad\text{for } j,k\in\mathbb{Z},\ j\neq j'. \tag{3.129}$$

Recall that (3.72) and Claim 2 give $x^{-\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2}=(a_2+j\varepsilon_2(\partial_2))v_{(i'-1)\varepsilon_1+j\varepsilon_2}$. Since

$$x^{-\varepsilon_1}\partial_2.x^{\pm\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=x^{\pm\varepsilon_3}\partial_2.x^{-\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.130}$$

we get

$$x^{-\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\quad\text{for } j,k\in\mathbb{Z},\ j\neq j' \tag{3.131}$$

by (3.117), (3.129) and induction on k.

As (3.116) and (3.119) show

$$\begin{aligned}x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}&=\frac{1}{\varepsilon_1(\partial_1)}\big[x^{\varepsilon_3}\partial_1,x^{\varepsilon_1}\partial_2\big].v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\\&=\big(a_2+j\varepsilon_2(\partial_2)\big)v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k+1)\varepsilon_3}\end{aligned}\tag{3.132}$$

for $j, k \in \mathbb{Z}$, using it and (3.119), (3.127) in $[x^{\varepsilon_1+\varepsilon_3}\partial_2, x^{\varepsilon_1}\partial_2].v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = 0$, we get

$$x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2+(k+1)\varepsilon_3} \quad \text{for } j, k \in \mathbb{Z},\ j \neq j'. \tag{3.133}$$

Inserting (3.116), (3.127) and (3.133) into

$$\big[x^{\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_2\big].v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_1(\partial_1)x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \tag{3.134}$$

and

$$\big[x^{-\varepsilon_3}\partial_1, x^{\varepsilon_1+\varepsilon_3}\partial_2\big].v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k-1)\varepsilon_3} = \varepsilon_1(\partial_1)x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k-1)\varepsilon_3} \tag{3.135}$$

respectively, we have

$$x^{\pm\varepsilon_3}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_1 + i'\varepsilon_1(\partial_1)\big)v_{i'\varepsilon_1+j\varepsilon_2+(k\pm1)\varepsilon_3} = 0 \quad \text{for } j, k \in \mathbb{Z},\ j \neq j'. \tag{3.136}$$

From (3.69) and Claim 2 we see that $x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2} = a_3 v_{i'\varepsilon_1+j\varepsilon_2}$ for $j \in \mathbb{Z}$. Inserting (3.117), (3.129) and (3.133) into

$$\big[x^{\varepsilon_1}\partial_3, x^{\varepsilon_3}\partial_2\big].v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_3(\partial_3)x^{\varepsilon_1+\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.137}$$

we find

$$x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j, k \in \mathbb{Z},\ j \neq j' \tag{3.138}$$

by induction on k.

Applying (3.117), (3.122), (3.132) and (3.133) to

$$\big[x^{-\varepsilon_1}\partial_3, x^{\varepsilon_1+\varepsilon_3}\partial_2\big].v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k-1)\varepsilon_3} = \varepsilon_3(\partial_3)x^{\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j\varepsilon_2+(k-1)\varepsilon_3}, \tag{3.139}$$

we obtain

$$x^{-\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j, k \in \mathbb{Z},\ j \neq j'. \tag{3.140}$$

From using (3.118) and (3.138) in

$$x^{\pm\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = x^{\varepsilon_1}\partial_3.x^{\pm\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.141}$$

it can be deduced

$$x^{\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}$$
$$\text{for } j \in \mathbb{Z}\backslash\big\{j', j'-1\big\},\ k \in \mathbb{Z}\backslash\big\{k'\big\}, \tag{3.142}$$
$$x^{-\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j-1)\varepsilon_2+k\varepsilon_3}$$
$$\text{for } j \in \mathbb{Z}\backslash\big\{j', j'+1\big\},\ k \in \mathbb{Z}\backslash\big\{k'\big\}. \tag{3.143}$$

On the other hand, from applying (3.129), (3.142) and (3.143) to

$$x^{\pm\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3}=\frac{1}{\varepsilon_2^2(\partial_2)}\big[\big[x^{\pm\varepsilon_2}\partial_3,x^{\varepsilon_3}\partial_2\big],x^{-\varepsilon_3}\partial_2\big].v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3},\tag{3.144}$$

it can be derived

$$x^{\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3}=0=\big(a_3+k'\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k'\varepsilon_3}\quad\text{for } j\in\mathbb{Z}\backslash\big\{j',j'-1\big\},\tag{3.145}$$

$$x^{-\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3}=0=\big(a_3+k'\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j-1)\varepsilon_2+k'\varepsilon_3}\quad\text{for } j\in\mathbb{Z}\backslash\big\{j',j'+1\big\},\tag{3.146}$$

which coincide with (3.142) and (3.143), respectively.

Using (3.136) in

$$x^{\pm\varepsilon_2}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\pm\frac{1}{\varepsilon_2(\partial_2)\varepsilon_3(\partial_3)}\big[\big[x^{\pm\varepsilon_2}\partial_3,x^{\varepsilon_3}\partial_2\big],x^{-\varepsilon_3}\partial_1\big].v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3},\tag{3.147}$$

we have

$$x^{\varepsilon_2}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=0=\big(a_1+i'\varepsilon_1(\partial_1)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}\quad\text{for } j,k\in\mathbb{Z},\ j\neq j',j'-1,\tag{3.148}$$

$$x^{-\varepsilon_2}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=0=\big(a_1+i'\varepsilon_1(\partial_1)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}\quad\text{for } j,k\in\mathbb{Z},\ j\neq j',j'+1.\tag{3.149}$$

Thirdly, under the condition that j' does exist, we define $\{v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\mid k\in\mathbb{Z}\backslash\{0,k'\}\}$, and determine

(1) the action of $x^{\varepsilon_1}\partial_2$ and $x^{\varepsilon_1}\partial_3$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k'\}}M_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$,
(2) the action of $x^{-\varepsilon_1}\partial_2$ and $x^{-\varepsilon_1}\partial_3$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k'\}}M_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$,
(3) the action of $x^{\varepsilon_2}\partial_1$ and $x^{\varepsilon_2}\partial_3$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k'\}}M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$ for $j=j',j'-1$,
(4) the action of $x^{-\varepsilon_2}\partial_1$ and $x^{-\varepsilon_2}\partial_3$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k'\}}M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$ for $j=j',j'+1$,
(5) the action of $x^{\varepsilon_3}\partial_1$ and $x^{\varepsilon_3}\partial_2$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k',k'-1\}}M_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$,
(6) the action of $x^{-\varepsilon_3}\partial_1$ and $x^{-\varepsilon_3}\partial_2$ on $\bigoplus_{k\in\mathbb{Z}\backslash\{k',k'+1\}}M_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$.

If j' does not exist, this part can be skipped.

Under the condition that j' exists, we want to define $v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$ for $k\in\mathbb{Z}\backslash\{0,k'\}$. As (3.118) and (3.138) show

$$x^{-\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=x^{\varepsilon_1}\partial_3.x^{-\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\neq 0\quad\text{for } k\in\mathbb{Z}\backslash\big\{k'\big\},\tag{3.150}$$

we have

$$x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\neq 0\quad\text{for } k\in\mathbb{Z}\backslash\big\{k'\big\}.\tag{3.151}$$

So we can define $v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}$'s by

$$x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\quad\text{for } k\in\mathbb{Z}\backslash\big\{0,k'\big\}.\tag{3.152}$$

This and (3.69) together with Claim 2 show

$$x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\quad \text{for } k\in\mathbb{Z}\backslash\big\{k'\big\}. \tag{3.153}$$

Next we shall derive the action of the other elements.
Applying (3.118), (3.138) and (3.153) to

$$x^{\pm\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=x^{\varepsilon_1}\partial_3.x^{\pm\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.154}$$

we obtain

$$x^{\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}$$
$$\text{for } j\in\mathbb{Z}\backslash\big\{j',j'-1\big\},\ k\in\mathbb{Z}\backslash\big\{k'\big\}, \tag{3.155}$$
$$x^{-\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j-1)\varepsilon_2+k\varepsilon_3}$$
$$\text{for } j\in\mathbb{Z}\backslash\big\{j',j'+1\big\},\ k\in\mathbb{Z}\backslash\big\{k'\big\}. \tag{3.156}$$

From using (3.118), (3.140) and (3.155) in

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_2}\partial_3.v_{i'\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}=x^{\varepsilon_2}\partial_3.x^{-\varepsilon_1}\partial_3.v_{i'\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}, \tag{3.157}$$

it can be deduced

$$x^{-\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}\quad \text{for } k\in\mathbb{Z}\backslash\big\{k'\big\}. \tag{3.158}$$

Since (3.115) and (3.120) indicate

$$\begin{aligned}x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}&=\frac{1}{\varepsilon_1(\partial_1)}\big[x^{\varepsilon_2}\partial_1,x^{\varepsilon_1}\partial_3\big].v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}\\&=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{(i'-1)\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}\end{aligned} \tag{3.159}$$

for $j,k\in\mathbb{Z}$, applying it and (3.120), (3.138), (3.153) to

$$x^{\varepsilon_1+\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_3.v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=x^{\varepsilon_1}\partial_3.x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{(i'-2)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.160}$$

we can derive

$$x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}=\big(a_3+k\varepsilon_3(\partial_3)\big)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3}$$
$$\text{for } j,k\in\mathbb{Z},\ k\neq k'. \tag{3.161}$$

Then using (3.118), (3.127), (3.155) and (3.161) in

$$\big[x^{\varepsilon_1}\partial_2,x^{\varepsilon_2}\partial_3\big].v_{(i'-1)\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}=\varepsilon_2(\partial_2)x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}, \tag{3.162}$$

we get

$$x^{\varepsilon_1}\partial_2.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=\big(a_2+j'\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}=0\quad \text{for } k\in\mathbb{Z}\backslash\big\{k'\big\}. \tag{3.163}$$

Moreover, using (3.118), (3.121), (3.159) and (3.161) in

$$\left[x^{-\varepsilon_1}\partial_2, x^{\varepsilon_1+\varepsilon_2}\partial_3\right].v_{(i'-1)\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3} = \varepsilon_2(\partial_2)x^{\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}, \tag{3.164}$$

we obtain

$$x^{-\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = \left(a_2 + j'\varepsilon_2(\partial_2)\right)v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 \quad \text{for } k \in \mathbb{Z}\backslash\left\{k'\right\}. \tag{3.165}$$

From inserting (3.115), (3.138), (3.153) and (3.161) into

$$\left[x^{\varepsilon_2}\partial_1, x^{\varepsilon_1}\partial_3\right].v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_1(\partial_1)x^{\varepsilon_1+\varepsilon_2}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.166}$$

$$\left[x^{-\varepsilon_2}\partial_1, x^{\varepsilon_1+\varepsilon_2}\partial_3\right].v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_1(\partial_1)x^{\varepsilon_1}\partial_3.v_{(i'-1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.167}$$

it can be respectively deduced

$$x^{\varepsilon_2}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \left(a_1 + i'\varepsilon_1(\partial_1)\right)v_{i'\varepsilon_1+(j+1)\varepsilon_2+k\varepsilon_3} = 0 \quad \text{for } j \in \left\{j', j'-1\right\},\ k \neq k', \tag{3.168}$$

$$x^{-\varepsilon_2}\partial_1.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \left(a_1 + i'\varepsilon_1(\partial_1)\right)v_{i'\varepsilon_1+(j-1)\varepsilon_2+k\varepsilon_3} = 0 \quad \text{for } j \in \left\{j', j'+1\right\},\ k \neq k'. \tag{3.169}$$

Substituting (3.117), (3.153) and (3.158) into

$$\left[x^{-\varepsilon_1}\partial_3, \left[x^{\varepsilon_1}\partial_3, x^{\pm\varepsilon_3}\partial_2\right]\right].v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = \varepsilon_3^2(\partial_3)x^{\pm\varepsilon_3}\partial_2.v_{(i'-1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3}, \tag{3.170}$$

we get

$$x^{\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \left(a_2 + j'\varepsilon_2(\partial_2)\right)v_{i'\varepsilon_1+j'\varepsilon_2+(k+1)\varepsilon_3} \quad \text{for } k \in \mathbb{Z}\backslash\left\{k', k'-1\right\}, \tag{3.171}$$

$$x^{-\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \left(a_2 + j'\varepsilon_2(\partial_2)\right)v_{i'\varepsilon_1+j'\varepsilon_2+(k-1)\varepsilon_3} \quad \text{for } k \in \mathbb{Z}\backslash\left\{k', k'+1\right\}. \tag{3.172}$$

Applying (3.136), (3.155) and (3.156) to

$$\left[x^{-\varepsilon_2}\partial_3, \left[x^{\varepsilon_2}\partial_3, x^{\pm\varepsilon_3}\partial_1\right]\right].v_{i'\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3} = \varepsilon_3^2(\partial_3)x^{\pm\varepsilon_3}\partial_1.v_{i'\varepsilon_1+(j'-1)\varepsilon_2+k\varepsilon_3}, \tag{3.173}$$

we obtain

$$x^{\varepsilon_3}\partial_1.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \left(a_1 + i'\varepsilon_1(\partial_1)\right)v_{i'\varepsilon_1+j'\varepsilon_2+(k+1)\varepsilon_3} \quad \text{for } k \in \mathbb{Z}\backslash\left\{k', k'-1\right\}, \tag{3.174}$$

$$x^{-\varepsilon_3}\partial_1.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \left(a_1 + i'\varepsilon_1(\partial_1)\right)v_{i'\varepsilon_1+j'\varepsilon_2+(k-1)\varepsilon_3} \quad \text{for } k \in \mathbb{Z}\backslash\left\{k', k'+1\right\}. \tag{3.175}$$

Fourthly, we want to derive

(1) the action of $x^{\varepsilon_1}\partial_2$ and $x^{\varepsilon_1}\partial_3$ on $\bigoplus_{(j',k')\neq(j,k)\in\mathbb{Z}^2} M_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$,
(2) the action of $x^{-\varepsilon_1}\partial_2$ and $x^{-\varepsilon_1}\partial_3$ on $\bigoplus_{(j',k')\neq(j,k)\in\mathbb{Z}^2} M_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3}$.

Taking $i = i'+1$ in (3.115)–(3.118), we have

$$x^{\pm\varepsilon_3}\partial_1.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_1(\partial_1)v_{(i'+1)\varepsilon_1+j\varepsilon_2+(k\pm 1)\varepsilon_3}, \tag{3.176}$$

$$x^{\pm\varepsilon_2}\partial_1.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \varepsilon_1(\partial_1)v_{(i'+1)\varepsilon_1+(j\pm 1)\varepsilon_2+k\varepsilon_3}, \tag{3.177}$$

$$x^{\pm\varepsilon_3}\partial_2.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{(i'+1)\varepsilon_1+j\varepsilon_2+(k\pm 1)\varepsilon_3}, \tag{3.178}$$

$$x^{\pm\varepsilon_2}\partial_3.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{(i'+1)\varepsilon_1+(j\pm 1)\varepsilon_2+k\varepsilon_3} \tag{3.179}$$

for $j,k\in\mathbb{Z}$.

Recall that (3.72) and Claim 2 give $x^{\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2} = (a_2 + j\varepsilon_2(\partial_2))v_{(i'+1)\varepsilon_1+j\varepsilon_2}$ for $j\in\mathbb{Z}$. Since

$$x^{\varepsilon_1}\partial_2.x^{\pm\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = x^{\pm\varepsilon_3}\partial_2.x^{\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.180}$$

we obtain

$$x^{\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j,k\in\mathbb{Z},\ j\neq j' \tag{3.181}$$

by (3.129), (3.178) and induction on k. On the other hand, since (3.155), (3.156), (3.179) and (3.181) lead to

$$\begin{aligned} 0 &= \big[x^{-\varepsilon_2}\partial_3, \big[x^{\varepsilon_2}\partial_3, x^{\varepsilon_1}\partial_2\big]\big].v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} \\ &= x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} + x^{\varepsilon_1}\partial_2.x^{\varepsilon_2}\partial_3.x^{-\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} \end{aligned} \tag{3.182}$$

for $k\in\mathbb{Z}\backslash\{k'\}$, we find

$$x^{\varepsilon_1}\partial_2.v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \big(a_2 + j'\varepsilon_2(\partial_2)\big)v_{(i'+1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} \quad \text{for } k\in\mathbb{Z}\backslash\big\{k'\big\} \tag{3.183}$$

by (3.155), (3.156) and (3.179).

Similarly, in analogy with (3.181), we get

$$x^{-\varepsilon_1}\partial_2.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_2 + j\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j,k\in\mathbb{Z},\ j\neq j'. \tag{3.184}$$

By the fact $[x^{-\varepsilon_2}\partial_3, [x^{-\varepsilon_1}\partial_2, x^{\varepsilon_2}\partial_3]].v_{(i'+1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0$, we derive

$$x^{-\varepsilon_1}\partial_2.v_{(i'+1)\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} = 0 = \big(a_2 + j'\varepsilon_2(\partial_2)\big)v_{i'\varepsilon_1+j'\varepsilon_2+k\varepsilon_3} \quad \text{for } k\in\mathbb{Z}\backslash\big\{k'\big\}. \tag{3.185}$$

Recall that (3.68) and Claim 2 give $x^{\varepsilon_1}\partial_3.v_{i'\varepsilon_1+k\varepsilon_3} = (a_3 + k\varepsilon_3(\partial_3))v_{(i'+1)\varepsilon_1+k\varepsilon_3}$ for $k\in\mathbb{Z}$. Since

$$x^{\varepsilon_1}\partial_3.x^{\pm\varepsilon_2}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = x^{\pm\varepsilon_2}\partial_3.x^{\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3}, \tag{3.186}$$

we obtain

$$x^{\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \big(a_3 + k\varepsilon_3(\partial_3)\big)v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j,k\in\mathbb{Z},\ k\neq k' \tag{3.187}$$

by (3.142), (3.143), (3.155), (3.156), (3.179) and induction on j. Since (3.129), (3.178) and (3.187) lead to

$$\begin{aligned} 0 &= \left[x^{-\varepsilon_3}\partial_2, \left[x^{\varepsilon_3}\partial_2, x^{\varepsilon_1}\partial_3\right]\right].v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} \\ &= x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.x^{\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} + x^{\varepsilon_1}\partial_3.x^{\varepsilon_3}\partial_2.x^{-\varepsilon_3}\partial_2.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} \end{aligned} \tag{3.188}$$

for $j \in \mathbb{Z}\backslash\{j'\}$, it can be deduced from (3.129) and (3.178) that

$$x^{\varepsilon_1}\partial_3.v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} = 0 = \left(a_3 + k'\varepsilon_3(\partial_3)\right)v_{(i'+1)\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} \quad \text{for } j \in \mathbb{Z}\backslash\left\{j'\right\}. \tag{3.189}$$

Similarly, in analogy with (3.187), we get

$$x^{-\varepsilon_1}\partial_3.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k\varepsilon_3} = \left(a_3 + k\varepsilon_3(\partial_3)\right)v_{i'\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \quad \text{for } j, k \in \mathbb{Z},\ k \neq k', \tag{3.190}$$

and by the fact $[x^{-\varepsilon_3}\partial_2, [x^{-\varepsilon_1}\partial_3, x^{\varepsilon_3}\partial_2]].v_{(i'+1)\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} = 0$, we deduce

$$x^{-\varepsilon_1}\partial_3.v_{(i'+1)\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} = 0 = \left(a_3 + k'\varepsilon_3(\partial_3)\right)v_{i'\varepsilon_1+j\varepsilon_2+k'\varepsilon_3} \quad \text{for } j \in \mathbb{Z}\backslash\left\{j'\right\}. \tag{3.191}$$

Finally, we completely specify the action of $\mathcal{X}$ on M and give the conclusion of this case.

By now, we have determined $\{v_{i\varepsilon_1+j\varepsilon_2+k\varepsilon_3} \mid (i', j', k') \neq (i, j, k) \in \mathbb{Z}^3\}$ in this case. Define $\mu \in D^*$ by $\mu(\partial_1) = a_1$, $\mu(\partial_2) = a_2$ and $\mu(\partial_3) = a_3$. Write $\zeta = i'\varepsilon_1 + j'\varepsilon_2 + k'\varepsilon_3$ for short. Summing up the acting relations mentioned above, we can write them uniformly by

$$x^{\epsilon}\partial.v_{\beta} = (\beta + \mu)(\partial)v_{\beta+\epsilon} \tag{3.192}$$

for $\epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$, $\partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}$ and $\beta \in \Gamma\backslash\{\zeta, \zeta - \epsilon\}$.

If j' or k' does not exist, i.e., ζ does not exist, then (3.192) shows the general action of $\mathcal{X}$. Since $\mathcal{X}$ generates $\mathcal{S}(\Gamma, D)$ (cf. (3.40)), we deduce

$$M \simeq \mathscr{M}_{\mu}. \tag{3.193}$$

If both j' and k' exist, then $\zeta = -\mu$ and we have a few acting relations left to be specified, say,

$$\begin{aligned} &\text{how } x^{\epsilon}\partial \text{ acts on } M_{\zeta} \text{ and on } M_{\zeta-\epsilon} \text{ for } \epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\} \\ &\quad \text{and } \partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}. \end{aligned} \tag{3.194}$$

The action in (3.194) is divided into three subcases.

The first subcase is that $x^{\epsilon}\partial.M_{\zeta} = \{0\}$ and $x^{\epsilon}\partial.v_{\zeta-\epsilon} = 0$ for all $\epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$ and $\partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}$. This and (3.192) give the general action of $\mathcal{X}$. Since $\mathcal{X}$ generates $\mathcal{S}(\Gamma, D)$ (cf. (3.40)), we deduce

$$M \simeq \mathscr{M}_{\mu}. \tag{3.195}$$

The second subcase is that $x^{\epsilon}\partial.M_{\zeta} \neq \{0\}$ for some $\epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$ and some $\partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}$. Assume $x^{\varepsilon_3}\partial_1.M_{\zeta} \neq \{0\}$; the other cases can be proved similarly. Pick a nonzero vector v_{ζ} of M_{ζ}. Write

$$x^{\varepsilon_3}\partial_1.v_{\zeta} = \hat{a}_1 v_{\zeta+\varepsilon_3}, \qquad x^{\varepsilon_1}\partial_2.v_{\zeta} = \hat{a}_2 v_{\zeta+\varepsilon_1}, \qquad x^{\varepsilon_2}\partial_3.v_{\zeta} = \hat{a}_3 v_{\zeta+\varepsilon_2} \tag{3.196}$$

with $\hat{a}_1, \hat{a}_2, \hat{a}_3 \in \mathbb{F}$, where $\hat{a}_1 \neq 0$.

By (3.192), we have

$$x^{\varepsilon_3}\partial_1.\big(x^{-\varepsilon_3}\partial_l.v_{\zeta+\varepsilon_3}\big)=x^{-\varepsilon_3}\partial_l.\big(x^{\varepsilon_3}\partial_1.v_{\zeta+\varepsilon_3}\big)=0 \quad \text{for } l\in\{1,2\}, \tag{3.197}$$

which implies

$$x^{-\varepsilon_3}\partial_l.v_{\zeta+\varepsilon_3}=0 \quad \text{for } l\in\{1,2\}. \tag{3.198}$$

Moreover, (3.192) yields

$$x^{\varepsilon_3}\partial_1.\big(x^{\pm\varepsilon_2}\partial_1.v_{\zeta\mp\varepsilon_2}\big)=x^{\pm\varepsilon_2}\partial_1.\big(x^{\varepsilon_3}\partial_1.v_{\zeta\mp\varepsilon_2}\big)=0, \tag{3.199}$$

which shows

$$x^{\pm\varepsilon_2}\partial_1.v_{\zeta\mp\varepsilon_2}=0. \tag{3.200}$$

Since (3.198) leads to

$$\big[x^{-\varepsilon_1}\partial_2,\big[x^{-\varepsilon_3}\partial_1,x^{\varepsilon_1}\partial_3\big]\big].v_{\zeta+\varepsilon_3}=\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)x^{-\varepsilon_3}\partial_2.v_{\zeta+\varepsilon_3}=0, \tag{3.201}$$

from (3.192) and (3.198) we derive

$$\begin{aligned} x^{-\varepsilon_1}\partial_2.v_{\zeta+\varepsilon_1} &= \frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}x^{-\varepsilon_1}\partial_2.\big(x^{-\varepsilon_3}\partial_1.x^{\varepsilon_1}\partial_3.v_{\zeta+\varepsilon_3}\big)\\ &= \frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_1}\partial_2,\big[x^{-\varepsilon_3}\partial_1,x^{\varepsilon_1}\partial_3\big]\big].v_{\zeta+\varepsilon_3}=0. \end{aligned} \tag{3.202}$$

Similarly, from

$$\big[x^{\varepsilon_1}\partial_2,\big[x^{-\varepsilon_3}\partial_1,x^{-\varepsilon_1}\partial_3\big]\big].v_{\zeta+\varepsilon_3}=-\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)x^{-\varepsilon_3}\partial_2.v_{\zeta+\varepsilon_3}=0 \tag{3.203}$$

it can be deduced

$$x^{\varepsilon_1}\partial_2.v_{\zeta-\varepsilon_1}=0. \tag{3.204}$$

Since

$$\begin{aligned} x^{\varepsilon_2}\partial_1.v_\zeta &= \frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big[x^{-\varepsilon_3}\partial_2,\big[x^{\varepsilon_2}\partial_3,x^{\varepsilon_3}\partial_1\big]\big].v_\zeta\\ &= \frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big(\hat{a}_1\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)v_{\zeta+\varepsilon_2}-x^{\varepsilon_2}\partial_3.x^{-\varepsilon_3}\partial_2.\big(x^{\varepsilon_3}\partial_1.v_\zeta\big)\\ &\quad +x^{\varepsilon_3}\partial_1.\big(x^{\varepsilon_2}\partial_3.x^{-\varepsilon_3}\partial_2.v_\zeta\big)\big)\\ &= \hat{a}_1v_{\zeta+\varepsilon_2}\neq 0 \end{aligned} \tag{3.205}$$

and

$$x^{\varepsilon_2}\partial_1.\big(x^{\pm\varepsilon_2}\partial_3.v_{\zeta\mp\varepsilon_2}\big)=x^{\pm\varepsilon_2}\partial_3.\big(x^{\varepsilon_2}\partial_1.v_{\zeta\mp\varepsilon_2}\big)=0, \tag{3.206}$$

we have

$$x^{\pm\varepsilon_2}\partial_3.v_{\zeta\mp\varepsilon_2}=0. \tag{3.207}$$

Moreover,

$$x^{-\varepsilon_3}\partial_1.v_\zeta=-\frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big[x^{-\varepsilon_2}\partial_3,\big[x^{-\varepsilon_3}\partial_2,x^{\varepsilon_2}\partial_1\big]\big].v_\zeta=\hat{a}_1v_{\zeta-\varepsilon_3}\neq 0 \tag{3.208}$$

and

$$x^{-\varepsilon_3}\partial_1.\big(x^{\varepsilon_3}\partial_l.v_{\zeta-\varepsilon_3}\big)=x^{\varepsilon_3}\partial_l.\big(x^{-\varepsilon_3}\partial_1.v_{\zeta-\varepsilon_3}\big)=0 \quad \text{for } l\in\{1,2\} \tag{3.209}$$

indicate

$$x^{\varepsilon_3}\partial_l.v_{\zeta-\varepsilon_3}=0 \quad \text{for } l\in\{1,2\}. \tag{3.210}$$

From

$$\big[x^{\pm\varepsilon_1}\partial_3,\big[x^{-\varepsilon_2}\partial_1,x^{\mp\varepsilon_1}\partial_2\big]\big].v_{\zeta+\varepsilon_2}=\mp\varepsilon_2(\partial_2)\varepsilon_1(\partial_1)x^{-\varepsilon_2}\partial_3.v_{\zeta+\varepsilon_2}=0 \tag{3.211}$$

it can be derived

$$x^{\pm\varepsilon_1}\partial_3.v_{\zeta\mp\varepsilon_1}=0. \tag{3.212}$$

So we have

$$x^{\epsilon}\partial.v_{\zeta-\epsilon}=0 \quad \text{for all } \epsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\} \text{ and } \partial\in\ker\epsilon\cap\{\partial_1,\partial_2,\partial_3\}. \tag{3.213}$$

Thereby we get

$$x^{-\varepsilon_2}\partial_1.v_\zeta=-\frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big[x^{-\varepsilon_3}\partial_2,\big[x^{-\varepsilon_2}\partial_3,x^{\varepsilon_3}\partial_1\big]\big].v_\zeta=\hat{a}_1v_{\zeta-\varepsilon_2}, \tag{3.214}$$

$$x^{\pm\varepsilon_3}\partial_2.v_\zeta=\pm\frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_1}\partial_3,\big[x^{\pm\varepsilon_3}\partial_1,x^{\varepsilon_1}\partial_2\big]\big].v_\zeta=\hat{a}_2v_{\zeta\pm\varepsilon_3}, \tag{3.215}$$

$$x^{-\varepsilon_1}\partial_2.v_\zeta=-\frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_3}\partial_1,\big[x^{-\varepsilon_1}\partial_3,x^{\varepsilon_3}\partial_2\big]\big].v_\zeta=\hat{a}_2v_{\zeta-\varepsilon_1}, \tag{3.216}$$

$$x^{\pm\varepsilon_1}\partial_3.v_\zeta=\pm\frac{1}{\varepsilon_2(\partial_2)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_2}\partial_1,\big[x^{\pm\varepsilon_1}\partial_2,x^{\varepsilon_2}\partial_3\big]\big].v_\zeta=\hat{a}_3v_{\zeta\pm\varepsilon_1}, \tag{3.217}$$

$$x^{-\varepsilon_2}\partial_3.v_\zeta=-\frac{1}{\varepsilon_2(\partial_2)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_1}\partial_2,\big[x^{-\varepsilon_2}\partial_1,x^{\varepsilon_1}\partial_3\big]\big].v_\zeta=\hat{a}_3v_{\zeta-\varepsilon_2}. \tag{3.218}$$

Define $\eta\in D^*$ by $\eta(\partial_1)=\hat{a}_1$, $\eta(\partial_2)=\hat{a}_2$ and $\eta(\partial_3)=\hat{a}_3$. Then we can write (3.196), (3.205), (3.208) and (3.214)–(3.218) uniformly as

$$x^{\epsilon}\partial.v_\zeta=\eta(\partial)v_{\zeta+\epsilon} \quad \text{for all } \epsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\} \text{ and } \partial\in\ker\epsilon\cap\{\partial_1,\partial_2,\partial_3\}. \tag{3.219}$$

Relations (3.213) and (3.219) show (3.194). So we have completely specified the general action of $\mathcal{X}$ (cf. (3.192), (3.213), (3.219)). Since $\mathcal{X}$ generates $\mathcal{S}(\Gamma, D)$ (cf. (3.40)), with a shift of the indices, we deduce

$$M \simeq \mathscr{A}_\eta. \tag{3.220}$$

The third subcase is that

$$x^\epsilon \partial . M_\zeta = \{0\} \quad \text{for all } \epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\} \text{ and } \partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}, \tag{3.221}$$

while there exist some $\epsilon' \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$ and $\partial' \in \ker\epsilon' \cap \{\partial_1, \partial_2, \partial_3\}$ such that

$$x^{\epsilon'} \partial' . v_{\zeta - \epsilon'} \neq 0. \tag{3.222}$$

Pick a nonzero vector v_ζ of M_ζ. Write

$$x^{\varepsilon_3}\partial_1 . v_{\zeta-\varepsilon_3} = \hat{a}'_1 v_\zeta, \qquad x^{\varepsilon_1}\partial_2 . v_{\zeta-\varepsilon_1} = \hat{a}'_2 v_\zeta, \qquad x^{\varepsilon_2}\partial_3 . v_{\zeta-\varepsilon_2} = \hat{a}'_3 v_\zeta \tag{3.223}$$

with $\hat{a}'_1, \hat{a}'_2, \hat{a}'_3 \in \mathbb{F}$. Then we have

$$x^{\pm\varepsilon_2}\partial_1 . v_{\zeta\mp\varepsilon_2} = \pm\frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big[x^{-\varepsilon_3}\partial_2, \big[x^{\pm\varepsilon_2}\partial_3, x^{\varepsilon_3}\partial_1\big]\big] . v_{\zeta\mp\varepsilon_2} = \hat{a}'_1 v_\zeta, \tag{3.224}$$

$$x^{-\varepsilon_3}\partial_1 . v_{\zeta+\varepsilon_3} = -\frac{1}{\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)}\big[x^{-\varepsilon_2}\partial_3, \big[x^{-\varepsilon_3}\partial_2, x^{\varepsilon_2}\partial_1\big]\big] . v_{\zeta+\varepsilon_3} = \hat{a}'_1 v_\zeta, \tag{3.225}$$

$$x^{\pm\varepsilon_3}\partial_2 . v_{\zeta\mp\varepsilon_3} = \pm\frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_1}\partial_3, \big[x^{\pm\varepsilon_3}\partial_1, x^{\varepsilon_1}\partial_2\big]\big] . v_{\zeta\mp\varepsilon_3} = \hat{a}'_2 v_\zeta, \tag{3.226}$$

$$x^{-\varepsilon_1}\partial_2 . v_{\zeta+\varepsilon_1} = -\frac{1}{\varepsilon_3(\partial_3)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_3}\partial_1, \big[x^{-\varepsilon_1}\partial_3, x^{\varepsilon_3}\partial_2\big]\big] . v_{\zeta+\varepsilon_1} = \hat{a}'_2 v_\zeta, \tag{3.227}$$

$$x^{\pm\varepsilon_1}\partial_3 . v_{\zeta\mp\varepsilon_1} = \pm\frac{1}{\varepsilon_2(\partial_2)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_2}\partial_1, \big[x^{\pm\varepsilon_1}\partial_2, x^{\varepsilon_2}\partial_3\big]\big] . v_{\zeta\mp\varepsilon_1} = \hat{a}'_3 v_\zeta, \tag{3.228}$$

$$x^{-\varepsilon_2}\partial_3 . v_{\zeta+\varepsilon_2} = -\frac{1}{\varepsilon_2(\partial_2)\varepsilon_1(\partial_1)}\big[x^{-\varepsilon_1}\partial_2, \big[x^{-\varepsilon_2}\partial_1, x^{\varepsilon_1}\partial_3\big]\big] . v_{\zeta+\varepsilon_2} = \hat{a}'_3 v_\zeta. \tag{3.229}$$

Define $\eta \in D^*$ by $\eta(\partial_1) = \hat{a}'_1$, $\eta(\partial_2) = \hat{a}'_2$ and $\eta(\partial_3) = \hat{a}'_3$. Then we can write (3.223)–(3.229) uniformly as

$$x^\epsilon \partial . v_{\zeta-\epsilon} = \eta(\partial) v_\zeta \quad \text{for all } \epsilon \in \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\} \text{ and } \partial \in \ker\epsilon \cap \{\partial_1, \partial_2, \partial_3\}. \tag{3.230}$$

Relations (3.221) and (3.230) show (3.194). So we have completely specified the general action of $\mathcal{X}$ (cf. (3.192), (3.221), (3.230)). Since $\mathcal{X}$ generates $\mathcal{S}(\Gamma, D)$ (cf. (3.40)), with a shift of the indices, we deduce

$$M \simeq \mathscr{B}_\eta. \tag{3.231}$$

Thus we complete the proof of Lemma 3.3. $\square$

Lemma 3.4. *Suppose the conditions of Lemma* 3.3 *fail. Moreover, if there exists some $\nu \in \Gamma$ such that*

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\nu = 0 \quad \textit{and} \quad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_\nu = 0,$$

or,

$$x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_\nu = 0 \quad \textit{and} \quad x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_\nu = 0,$$

or,

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\nu = 0 \quad \textit{and} \quad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_\nu = 0,$$

then $M \simeq \bigoplus_{\theta\in\Gamma} \mathbb{F}w_\theta$, where each component is a trivial submodule of $\mathcal{S}(\Gamma, D)$.

Proof. Suppose there exists some $\nu \in \Gamma$ such that $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\nu = 0$ and $x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_\nu = 0$; the other cases can be proved similarly. By a translation of the indices if necessary, we may assume $\nu = 0$. In other words,

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0 = 0 \quad \text{and} \quad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0 = 0. \tag{3.232}$$

Lemma 3.1 then implies

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_{k\varepsilon_3} = 0 \quad \text{and} \quad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_{k\varepsilon_3} = 0 \quad \text{for } k \in \mathbb{Z}. \tag{3.233}$$

We begin with the following claim:

Claim. *$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0 = 0$. Moreover, either $x^{\varepsilon_2}\partial_1.w_0 = 0$ or $x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_{\varepsilon_2} \neq 0$.*

If $x^{\varepsilon_2}\partial_1.w_0 = 0$, this claim holds trivially.
Suppose $x^{\varepsilon_2}\partial_1.w_0 \neq 0$. We proceed our proof in three cases.

Case 1. $x^{\varepsilon_3}\partial_1.w_0 = 0$.

Since $x^{\varepsilon_2}\partial_1.w_0 \neq 0$ and $x^{\varepsilon_3}\partial_1.(x^{\varepsilon_2}\partial_1.w_0) = x^{\varepsilon_2}\partial_1.x^{\varepsilon_3}\partial_1.w_0 = 0$, we have $x^{\varepsilon_3}\partial_1.w_{\varepsilon_2} = 0$, which further implies

$$\begin{aligned} x^{\varepsilon_2+\varepsilon_3}\partial_1.w_0 &= \frac{1}{\varepsilon_3(\partial_3)}\big[x^{\varepsilon_2}\partial_3, x^{\varepsilon_3}\partial_1\big].w_0 \\ &= \frac{1}{\varepsilon_3(\partial_3)}\big(x^{\varepsilon_2}\partial_3.x^{\varepsilon_3}\partial_1.w_0 - x^{\varepsilon_3}\partial_1.\big(x^{\varepsilon_2}\partial_3.w_0\big)\big) = 0. \end{aligned} \tag{3.234}$$

Thus

$$x^{\varepsilon_2+\varepsilon_3}\partial_1.x^{-\varepsilon_3}\partial_2.w_0 = \big[x^{\varepsilon_2+\varepsilon_3}\partial_1, x^{-\varepsilon_3}\partial_2\big].w_0 = -\varepsilon_2(\partial_2)x^{\varepsilon_2}\partial_1.w_0 \neq 0, \tag{3.235}$$

which indicates

$$x^{-\varepsilon_3}\partial_2.w_0 \neq 0 \quad \text{and} \quad x^{\varepsilon_2+\varepsilon_3}\partial_1.w_{-\varepsilon_3} \neq 0. \tag{3.236}$$

Since (3.232) shows

$$x^{\varepsilon_3}\partial_2.\big(x^{-\varepsilon_3}\partial_2.w_0\big) = x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0 = 0, \tag{3.237}$$

we get $x^{\varepsilon_3}\partial_2.w_{-\varepsilon_3} = 0$ by (3.236). Then

$$x^{\varepsilon_3}\partial_2.x^{\varepsilon_2}\partial_1.w_{-\varepsilon_3} = \big[x^{\varepsilon_3}\partial_2, x^{\varepsilon_2}\partial_1\big].w_{-\varepsilon_3} = \varepsilon_2(\partial_2)x^{\varepsilon_2+\varepsilon_3}\partial_1.w_{-\varepsilon_3} \neq 0, \tag{3.238}$$

which implies $x^{\varepsilon_2}\partial_1.w_{-\varepsilon_3} \neq 0$.

If $x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} = 0$, then

$$x^{\varepsilon_3}\partial_1.\big(x^{\varepsilon_2}\partial_1.w_{-\varepsilon_3}\big) = x^{\varepsilon_2}\partial_1.x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} = 0, \tag{3.239}$$

which leads to $x^{\varepsilon_3}\partial_1.w_{\varepsilon_2-\varepsilon_3} = 0$. Thus

$$\begin{aligned} x^{\varepsilon_2+\varepsilon_3}\partial_1.w_{-\varepsilon_3} &= \frac{1}{\varepsilon_3(\partial_3)}\big[x^{\varepsilon_2}\partial_3, x^{\varepsilon_3}\partial_1\big].w_{-\varepsilon_3} \\ &= \frac{1}{\varepsilon_3(\partial_3)}\big(x^{\varepsilon_2}\partial_3.x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} - x^{\varepsilon_3}\partial_1.\big(x^{\varepsilon_2}\partial_3.w_{-\varepsilon_3}\big)\big) = 0, \end{aligned} \tag{3.240}$$

which contradicts (3.236).

Assume $x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} \neq 0$. Then (3.236) and $x^{\varepsilon_3}\partial_1.w_0 = 0$ give

$$0 = x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_1.w_0 = x^{\varepsilon_3}\partial_1.\big(x^{-\varepsilon_3}\partial_2.w_0\big) \neq 0, \tag{3.241}$$

which is a contradiction.

Case 2. $x^{-\varepsilon_3}\partial_1.w_0 = 0$.

Replacing ε_3 by $-\varepsilon_3$ in the arguments of Case 1, we similarly get a contradiction.

Case 3. $x^{\varepsilon_3}\partial_1.w_0 \neq 0$ and $x^{-\varepsilon_3}\partial_1.w_0 \neq 0$.

Eq. (3.232) shows

$$x^{-\varepsilon_3}\partial_1.\big(x^{\varepsilon_3}\partial_1.w_0\big) = x^{\varepsilon_3}\partial_1.\big(x^{-\varepsilon_3}\partial_1.w_0\big) = 0, \tag{3.242}$$

which indicates $x^{-\varepsilon_3}\partial_1.w_{\varepsilon_3} = 0$ and $x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} = 0$.

If $x^{\varepsilon_2}\partial_1.w_{-\varepsilon_3} \neq 0$, in analogy with Case 1, $x^{\varepsilon_3}\partial_1.w_{-\varepsilon_3} = 0$ leads to a contradiction.

If $x^{\varepsilon_2}\partial_1.w_{\varepsilon_3} \neq 0$, in analogy with Case 2, $x^{-\varepsilon_3}\partial_1.w_{\varepsilon_3} = 0$ leads to a contradiction.

Assume $x^{\varepsilon_2}\partial_1.w_{\pm\varepsilon_3} = 0$. Then we have

$$x^{\varepsilon_3}\partial_1.\big(x^{\varepsilon_2}\partial_1.w_0\big) = x^{\varepsilon_2}\partial_1.\big(x^{\varepsilon_3}\partial_1.w_0\big) = 0. \tag{3.243}$$

Since $x^{\varepsilon_2}\partial_1.w_0 \neq 0$, (3.243) shows $x^{\varepsilon_3}\partial_1.w_{\varepsilon_2} = 0$, which further shows

$$x^{\varepsilon_3}\partial_1.\big(x^{-\varepsilon_2}\partial_1.w_{\varepsilon_2}\big) = x^{-\varepsilon_2}\partial_1.x^{\varepsilon_3}\partial_1.w_{\varepsilon_2} = 0. \tag{3.244}$$

Since $x^{\varepsilon_3}\partial_1.w_0 \neq 0$, we get $x^{-\varepsilon_2}\partial_1.w_{\varepsilon_2} = 0$ by (3.244). So we have

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0=0, \tag{3.245}$$

which completes the proof of the first statement of the claim. Moreover,

$$0\neq \varepsilon_2^2(\partial_2)x^{\varepsilon_2}\partial_1.w_0=\big[x^{\varepsilon_3}\partial_2,\big[x^{-\varepsilon_3}\partial_2,x^{\varepsilon_2}\partial_1\big]\big].w_0=x^{\varepsilon_3}\partial_2.x^{-\varepsilon_3}\partial_2.x^{\varepsilon_2}\partial_1.w_0 \tag{3.246}$$

gives

$$x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_{\varepsilon_2}\neq 0. \tag{3.247}$$

Thus this claim follows.

By symmetry, we can similarly prove $x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_0=0$.

So we conclude that $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0=0$ and $x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0=0$ lead to

$$x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0=0 \quad\text{and}\quad x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_0=0. \tag{3.248}$$

On the other hand, as the conditions of Lemma 3.3 fail, we must have

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_0=0 \quad\text{or}\quad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_0=0. \tag{3.249}$$

Without loss of generality, we assume $x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_0=0$. Observe that the above claim gives $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0=0$. Thus, in analogy with (3.248), from $x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_0=0$ and $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0=0$ we can deduce

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_0=0. \tag{3.250}$$

Therefore we obtain

$$x^{-\varepsilon}\partial.x^{\varepsilon}\partial.w_0=0 \quad \text{for all } \varepsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\} \text{ and } \partial\in\ker\varepsilon\cap\{\partial_1,\partial_2,\partial_3\}. \tag{3.251}$$

Lemma 3.1 and the above discussion then further imply

$$x^{-\varepsilon}\partial.x^{\varepsilon}\partial.w_\theta=0 \quad \text{for all } \varepsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\},\ \partial\in\ker\varepsilon\cap\{\partial_1,\partial_2,\partial_3\} \text{ and } \theta\in\Gamma. \tag{3.252}$$

Since the above claim states either $x^{\varepsilon_2}\partial_1.w_0=0$ or $x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_{\varepsilon_2}\neq 0$, (3.252) confirms

$$x^{\varepsilon_2}\partial_1.w_0=0. \tag{3.253}$$

Likewise, it can be proved

$$x^{\varepsilon}\partial.w_0=0 \quad \text{for all } \varepsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\} \text{ and } \partial\in\ker\varepsilon\cap\{\partial_1,\partial_2,\partial_3\}. \tag{3.254}$$

Moreover, it can be similarly proved

$$x^{\varepsilon}\partial.w_\theta=0 \quad \text{for all } \varepsilon\in\{\pm\varepsilon_1,\pm\varepsilon_2,\pm\varepsilon_3\},\ \partial\in\ker\varepsilon\cap\{\partial_1,\partial_2,\partial_3\} \text{ and } \theta\in\Gamma. \tag{3.255}$$

Therefore, (3.255) shows $M\simeq\bigoplus_{\theta\in\Gamma}\mathbb{F}w_\theta$, where each component is a trivial submodule of $\mathcal{S}(\Gamma,D)$. $\square$

Lemma 3.5. *The conditions of Lemma* 3.3 *and that of Lemma* 3.4 *enumerate all the possibilities.*

Proof. Suppose otherwise, namely, both the conditions of Lemma 3.3 and that of Lemma 3.4 fail. Then this will lead to a contradiction. The following proof is divided into two cases.

Case 1. $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0=0$.

Since both the conditions of Lemma 3.3 and that of Lemma 3.4 fail, we get

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0=0,\qquad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0\neq 0,\qquad x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_0=0,\tag{3.256}$$

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_0\neq 0,\qquad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_0=0,\qquad x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0\neq 0.\tag{3.257}$$

Lemma 3.1 then implies, for any $\theta\in\Gamma$,

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\theta=0,\qquad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_\theta\neq 0,\qquad x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_\theta=0,\tag{3.258}$$

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_\theta\neq 0,\qquad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_\theta=0,\qquad x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\theta\neq 0.\tag{3.259}$$

Since $x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\theta\neq 0$ for any $\theta\in\Gamma$, we get $x^{\pm\varepsilon_2}\partial_1.w_\theta\neq 0$ for any $\theta\in\Gamma$. Similarly, $x^{\pm\varepsilon_3}\partial_2.w_\theta\neq 0$ for any $\theta\in\Gamma$.

If $x^{\varepsilon_3}\partial_1.w_0=0$, by

$$x^{\varepsilon_3}\partial_1.\big(x^{\pm\varepsilon_2}\partial_1.w_{j\varepsilon_2}\big)=x^{\pm\varepsilon_2}\partial_1.x^{\varepsilon_3}\partial_1.w_{j\varepsilon_2}\tag{3.260}$$

and induction on j, we get $x^{\varepsilon_3}\partial_1.w_{j\varepsilon_2}=0$ for all $j\in\mathbb{Z}$. Then by the fact

$$x^{\varepsilon_3}\partial_1.\big(x^{\pm\varepsilon_3}\partial_2.w_{j\varepsilon_2+k\varepsilon_3}\big)=x^{\pm\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_1.w_{j\varepsilon_2+k\varepsilon_3}\tag{3.261}$$

and induction on k, we obtain $x^{\varepsilon_3}\partial_1.w_{j\varepsilon_2+k\varepsilon_3}=0$ for all $j,k\in\mathbb{Z}$. Therefore,

$$0\neq\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)x^{\varepsilon_2}\partial_1.w_0=\big[\big[x^{\varepsilon_3}\partial_1,x^{\varepsilon_2}\partial_3\big],x^{-\varepsilon_3}\partial_2\big].w_0=0\tag{3.262}$$

leads to a contradiction.

If $x^{\varepsilon_3}\partial_1.w_0\neq 0$, then we get $x^{-\varepsilon_3}\partial_1.w_{\varepsilon_3}=0$. By similar arguments as those from (3.260) to (3.261), we can prove

$$x^{-\varepsilon_3}\partial_1.w_{j\varepsilon_2+k\varepsilon_3}=0\quad\text{for all } j,k\in\mathbb{Z}.\tag{3.263}$$

So it can be deduced

$$0\neq-\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)x^{\varepsilon_2}\partial_1.w_0=\big[\big[x^{-\varepsilon_3}\partial_1,x^{\varepsilon_2}\partial_3\big],x^{\varepsilon_3}\partial_2\big].w_0=0,\tag{3.264}$$

which is a contradiction.

Case 2. $x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0\neq 0$.

Since both the conditions of Lemma 3.3 and that of Lemma 3.4 fail, we get

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_0 \neq 0, \qquad x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_0 = 0, \qquad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_0 \neq 0, \tag{3.265}$$

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_0 = 0, \qquad x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_0 \neq 0, \qquad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0 = 0. \tag{3.266}$$

Lemma 3.1 then implies, for any $\theta \in \Gamma$,

$$x^{-\varepsilon_3}\partial_1.x^{\varepsilon_3}\partial_1.w_\theta \neq 0, \qquad x^{-\varepsilon_2}\partial_1.x^{\varepsilon_2}\partial_1.w_\theta = 0, \qquad x^{-\varepsilon_2}\partial_3.x^{\varepsilon_2}\partial_3.w_\theta \neq 0, \tag{3.267}$$

$$x^{-\varepsilon_1}\partial_3.x^{\varepsilon_1}\partial_3.w_\theta = 0, \qquad x^{-\varepsilon_1}\partial_2.x^{\varepsilon_1}\partial_2.w_\theta \neq 0, \qquad x^{-\varepsilon_3}\partial_2.x^{\varepsilon_3}\partial_2.w_0 = 0. \tag{3.268}$$

Since

$$\big[\big[x^{\pm\varepsilon_2}\partial_1, x^{\varepsilon_3}\partial_2\big], x^{\mp\varepsilon_2}\partial_3\big].w_0 = \pm\varepsilon_3(\partial_3)\varepsilon_2(\partial_2)x^{\varepsilon_3}\partial_1.w_0, \tag{3.269}$$

similarly, in analogy with Case 1, we will get a contradiction. We omit the details. □

In summary, Lemmas 3.3–3.5, together with Lemma 2.1 and Theorem 2.2, give:

Lemma 3.6. *If* $\dim D = 3$ *and* $\Gamma \simeq \mathbb{Z}^3$, *then Theorem* 2.3 *holds.*

4. General case of dim $D = 3$

In this section, we deal with the general case of $\dim D = 3$, that is, we want to prove:

Lemma 4.1. *If* $\dim D = 3$, *then Theorem* 2.3 *holds.*

Proof. Assume that $M = \bigoplus_{\theta\in\Gamma} M_\theta$ is a Γ-graded $\mathcal{S}(\Gamma, D)$-module with $\dim M_\theta = 1$ for each $\theta \in \Gamma$. Suppose Γ' is an arbitrary subgroup of Γ such that $\bigcap_{\alpha\in\Gamma'} \ker\alpha = \{0\}$. Then $\mathcal{S}(\Gamma', D)$ is not only a subalgebra of $\mathcal{S}(\Gamma, D)$, but also a simple generalized divergence-free Lie algebra. Define

$$M\big(\nu, \Gamma'\big) = \bigoplus_{\theta\in\Gamma'} M_{\nu+\theta} \quad \text{for any } \nu \in \Gamma. \tag{4.1}$$

Then $M(\nu, \Gamma')$ is a Γ'-graded $\mathcal{S}(\Gamma', D)$-submodule.

By Zorn's lemma and Lemma 3.6, there exists a subgroup Γ_0 of Γ which is maximal with the property: $\bigcap_{\alpha\in\Gamma_0} \ker\alpha = \{0\}$ and the following condition holds:

(C1) Let $\{\theta_r \mid r \in I\}$ be the set of all representatives of cosets of Γ_0 in Γ. For any $r \in I$, $M(\theta_r, \Gamma_0)$ is isomorphic to:

$$\text{(i)} \quad \mathscr{M}_{\mu_r}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \mu_r \in D^*\backslash\Gamma_0; \tag{4.2}$$

$$\text{(ii)} \quad \mathscr{M}_{\mu_r}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \mu_r \in \Gamma_0; \tag{4.3}$$

$$\text{(iii)} \quad \mathscr{A}_{\eta_r}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \eta_r \in D^*\backslash\{0\}; \tag{4.4}$$

$$\text{(iv)} \quad \mathscr{B}_{\eta_r}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \eta_r \in D^*\backslash\{0\}; \tag{4.5}$$

$$\text{(v)} \quad \bigoplus_{\nu\in\Gamma_0} \mathbb{F}w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(\Gamma_0, D). \tag{4.6}$$

To prove this lemma, it suffices to show $\Gamma_0 = \Gamma$. Suppose $\Gamma_0 \neq \Gamma$. We will see that this leads to a contradiction. Picking $\theta_1 \in \Gamma \backslash \Gamma_0$, we set

$$\Gamma_1 = \Gamma_0 + \mathbb{Z}\theta_1. \tag{4.7}$$

Let

$$\{\rho_\lambda \mid \lambda \in J\} \text{ be the set of all representatives of cosets of } \Gamma_1 \text{ in } \Gamma. \tag{4.8}$$

Our intention is to prove that Γ_1 satisfies condition (C1), which results in the contradiction. Let

$$\{k\theta_1 \mid k \in K\} \text{ be the set of all representatives of cosets of } \Gamma_0 \text{ in } \Gamma_1, \text{ where } K \subseteq \mathbb{Z}. \tag{4.9}$$

Then (4.7) shows $\#(K) > 1$.

Fix $\lambda \in J$ hereafter. We want to see how $\mathcal{S}(\Gamma_1, D)$ acts on $M(\rho_\lambda, \Gamma_1)$ and therefore deduce Γ_1 satisfies condition (C1).

Condition (C1) implies:

(C2) For any $k \in \mathbb{Z}$, $M(\rho_\lambda + k\theta_1, \Gamma_0)$ is isomorphic to

$$\text{(i)} \quad \mathscr{M}_{\mu_k'}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \mu_k' \in D^* \backslash \Gamma_0; \tag{4.10}$$

$$\text{(ii)} \quad \mathscr{M}_{\mu_k'}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \mu_k' \in \Gamma_0; \tag{4.11}$$

$$\text{(iii)} \quad \mathscr{A}_{\eta_k'}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \eta_k' \in D^* \backslash \{0\}; \tag{4.12}$$

$$\text{(iv)} \quad \mathscr{B}_{\eta_k'}\big(\mathcal{S}(\Gamma_0, D)\big) \text{ for some } \eta_k' \in D^* \backslash \{0\}; \tag{4.13}$$

$$\text{(v)} \quad \bigoplus_{\nu \in \Gamma_0} \mathbb{F} w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(\Gamma_0, D). \tag{4.14}$$

In (4.10) and (4.11), μ_k''s are respectively chosen so that there exist nonzero $v_{\rho_\lambda + k\theta_1 + \beta} \in M_{\rho_\lambda + k\theta_1 + \beta}$ with $\beta \in \Gamma_0$ such that

$$x^\alpha \partial . v_{\rho_\lambda + k\theta_1 + \beta} = \big(\beta + \mu_k'\big)(\partial) v_{\rho_\lambda + k\theta_1 + \beta + \alpha} \tag{4.15}$$

for $\alpha \in \Gamma_0 \backslash \{0\}$, $\partial \in \ker \alpha$ and $\beta \in \Gamma_0$. If $M(\rho_\lambda + k\theta_1, \Gamma_0)$ is isomorphic to $\mathscr{A}_{\eta_k'}$ or $\mathscr{B}_{\eta_k'}$ (cf. (4.12), (4.13)), then there exists some $\mu_k' \in \Gamma_0$ such that, we can choose nonzero $v_{\rho_\lambda + k\theta_1 + \beta} \in M_{\rho_\lambda + k\theta_1 + \beta}$ with $\beta \in \Gamma_0$ so that

$$x^\alpha \partial . v_{\rho_\lambda + k\theta_1 + \beta} = \big(\beta + \mu_k'\big)(\partial) v_{\rho_\lambda + k\theta_1 + \beta + \alpha} \tag{4.16}$$

for $\alpha \in \Gamma_0 \backslash \{0\}$, $\partial \in \ker \alpha$ and $\beta \in \Gamma_0$ with $\beta + \mu_k' \neq 0 \neq \beta + \mu_k' + \alpha$.

Choose $\epsilon_1, \epsilon_2 \in \Gamma_0 \backslash \{0\}$ such that $\ker \theta_1 \cap \ker \epsilon_1 \cap \ker \epsilon_2 = \{0\}$. Set

$$G_0 = \mathbb{Z}\theta_1 + \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2. \tag{4.17}$$

Then $\mathcal{S}(G_0, D)$ is not only a subalgebra of $\mathcal{S}(\Gamma, D)$, but also a simple generalized divergence-free Lie algebra. By Lemma 3.6, we know that:

(C3) For any $\beta \in \Gamma_0$, $M(\rho_\lambda + \beta, G_0)$ is isomorphic to

$$\text{(i)} \quad \mathscr{M}_{\mu''_\beta}\big(\mathcal{S}(G_0, D)\big) \text{ for some } \mu''_\beta \in D^*\backslash G_0; \tag{4.18}$$

$$\text{(ii)} \quad \mathscr{M}_{\mu''_\beta}\big(\mathcal{S}(G_0, D)\big) \text{ for some } \mu''_\beta \in G_0; \tag{4.19}$$

$$\text{(iii)} \quad \mathscr{A}_{\eta''_\beta}\big(\mathcal{S}(G_0, D)\big) \text{ for some } \eta''_\beta \in D^*\backslash\{0\}; \tag{4.20}$$

$$\text{(iv)} \quad \mathscr{B}_{\eta''_\beta}\big(\mathcal{S}(G_0, D)\big) \text{ for some } \eta''_\beta \in D^*\backslash\{0\}; \tag{4.21}$$

$$\text{(v)} \quad \bigoplus_{\nu \in G_0} \mathbb{F} w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(G_0, D). \tag{4.22}$$

In (4.18) and (4.19), μ''_β's are respectively chosen so that, there exist nonzero $v_{\rho_\lambda+\beta+\nu} \in M_{\rho_\lambda+\beta+\nu}$ with $\nu \in G_0$ such that

$$x^\alpha \partial . v_{\rho_\lambda+\beta+\nu} = \big(\nu + \mu''_\beta\big)(\partial) v_{\rho_\lambda+\beta+\nu+\alpha} \tag{4.23}$$

for $\alpha \in G_0\backslash\{0\}$, $\partial \in \ker\alpha$ and $\nu \in G_0$. If $M(\rho_\lambda + \beta, G_0)$ is isomorphic to $\mathscr{A}_{\eta''_\beta}$ or $\mathscr{B}_{\eta''_\beta}$ (cf. (4.20), (4.21)), then there exists some $\mu''_\beta \in G_0$ such that, we can choose nonzero $v_{\rho_\lambda+\beta+\nu} \in M_{\rho_\lambda+\beta+\nu}$ with $\nu \in G_0$ so that

$$x^\alpha \partial . v_{\rho_\lambda+\beta+\nu} = \big(\nu + \mu''_\beta\big)(\partial) v_{\rho_\lambda+\beta+\nu+\alpha} \tag{4.24}$$

for $\alpha \in G_0\backslash\{0\}$, $\partial \in \ker\alpha$ and $\nu \in G_0$ with $\nu + \mu''_\beta \neq 0 \neq \nu + \mu''_\beta + \alpha$.

Our aim now is to show that Γ_1 satisfies condition (C1). The idea is to sew the $\mathcal{S}(\Gamma_0, D)$-submodules $M(\rho_\lambda + k\theta_1, \Gamma_0)$ for $k \in K$ together via the action of $\mathcal{S}(G_0, D)$. Before getting to that, we give some observations.

For any $\beta' \in \Gamma_1\backslash\Gamma_0$, we can choose $\theta'_1 \in (\theta_1 + \Gamma_0) \cap G_0$ and $\epsilon'_1, \epsilon'_2 \in \Gamma_0\backslash\{0\}$ such that

$$\ker\theta'_1 \cap \ker\epsilon'_1 \cap \ker\epsilon'_2 = \{0\} \quad \text{and} \quad \beta' \in G_1 = \mathbb{Z}\theta'_1 + \mathbb{Z}\epsilon'_1 + \mathbb{Z}\epsilon'_2. \tag{4.25}$$

Explicitly, we write $\beta' = k\theta_1 + \beta$ with $k \in K\backslash\{0\}$ and $\beta \in \Gamma_0$. If $\beta = 0$, we take $\theta'_1 = \theta_1$, and choose $\epsilon'_1, \epsilon'_2 \in \Gamma_0\backslash\{0\}$ such that $\ker\theta'_1 \cap \ker\epsilon'_1 \cap \ker\epsilon'_2 = \{0\}$. In the case $\beta \neq 0$ and $\ker\theta_1 \neq \ker\beta$, we take $\theta'_1 = \theta_1$, $\epsilon'_1 = \beta$, and choose $\epsilon'_2 \in \Gamma_0\backslash\{0\}$ such that $\ker\theta'_1 \cap \ker\epsilon'_1 \cap \ker\epsilon'_2 = \{0\}$. When $\ker\theta_1 = \ker\beta$, by the fact $\ker(\theta_1 + \epsilon_1) \neq \ker(\beta - k\epsilon_1)$ (cf. (4.17)), we can take $\theta'_1 = \theta_1 + \epsilon_1$, $\epsilon'_1 = \beta - k\epsilon_1$, and choose $\epsilon'_2 \in \Gamma_0\backslash\{0\}$ such that $\ker\theta'_1 \cap \ker\epsilon'_1 \cap \ker\epsilon'_2 = \{0\}$.

Notice that, for the above defined G_1, $\mathcal{S}(G_1, D)$ is not only a subalgebra of $\mathcal{S}(\Gamma, D)$, but also a simple generalized divergence-free Lie algebra.

Now we proceed our analysis through sewing the $\mathcal{S}(\Gamma_0, D)$-submodules $M(\rho_\lambda + k\theta_1, \Gamma_0)$ for $k \in K$ together. The proof is divided into five cases.

Case 1. There exists some $k_0 \in K$ such that $M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \bigoplus_{\nu \in \Gamma_0} \mathbb{F} w_\nu$, where each component is a trivial submodule of $\mathcal{S}(\Gamma_0, D)$.

Recall that $G_0 = \mathbb{Z}\theta_1 + \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2$, where $\epsilon_1, \epsilon_2 \in \Gamma_0\backslash\{0\}$. Since $M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \bigoplus_{\nu \in \Gamma_0} \mathbb{F} w_\nu$, where each component is a trivial submodule of $\mathcal{S}(\Gamma_0, D)$, we have

$$x^{\epsilon_s} \partial . M_{\rho_\lambda + k_0\theta_1 + \beta + l\epsilon_1 + m\epsilon_2} = \{0\}, \quad \forall \beta \in \Gamma_0,\ l, m \in \mathbb{Z},\ \partial \in \ker\epsilon_s \text{ and } s \in \{1, 2\}. \tag{4.26}$$

Pick $\partial' \in \ker\epsilon_1 \setminus \ker\epsilon_2$. In (C3), if $M(\rho_\lambda + \beta, G_0)$ for some $\beta \in \Gamma_0$ is isomorphic to one of the first four cases, then there exist $l', m' \in \mathbb{Z}$ such that

$$x^{\epsilon_1}\partial'.M_{\rho_\lambda+\beta+k_0\theta_1+l'\epsilon_1+m'\epsilon_2} \neq \{0\}, \tag{4.27}$$

which contradicts (4.26). So for any $\beta \in \Gamma_0$,

$$M(\rho_\lambda + \beta, G_0) \simeq \bigoplus_{\nu \in G_0} \mathbb{F}w_\nu, \tag{4.28}$$

where each component is a trivial submodule of $\mathcal{S}(G_0, D)$. Then we have

$$x^{\epsilon_s}\partial.M_{\rho_\lambda+\beta+k\theta_1+l\epsilon_1+m\epsilon_2} = \{0\}, \quad \forall \beta \in \Gamma_0,\ k, l, m \in \mathbb{Z},\ \partial \in \ker\epsilon_s \text{ and } s \in \{1, 2\}. \tag{4.29}$$

Pick $\partial' \in \ker\epsilon_1 \setminus \ker\epsilon_2$. Observe that in (C2), for any fixed $k \in K$, if $M(\rho_\lambda + k\theta_1, \Gamma_0)$ is isomorphic to one of the first four cases, then there exist $\beta \in \Gamma_0$ and $l', m' \in \mathbb{Z}$ such that

$$x^{\epsilon_1}\partial'.M_{\rho_\lambda+\beta+k\theta_1+l'\epsilon_1+m'\epsilon_2} \neq \{0\}, \tag{4.30}$$

which contradicts (4.29). So for any $k \in K$,

$$M(\rho_\lambda + k\theta_1, \Gamma_0) \simeq \bigoplus_{\nu \in \Gamma_0} \mathbb{F}w_\nu, \tag{4.31}$$

where each component is a trivial submodule of $\mathcal{S}(\Gamma_0, D)$. In other words, from (4.9) we see that

$$x^\alpha\partial.M(\rho_\lambda, \Gamma_1) = \{0\}, \quad \forall \alpha \in \Gamma_0 \setminus \{0\},\ \partial \in \ker\alpha. \tag{4.32}$$

Recall that for any $\beta' \in \Gamma_1 \setminus \Gamma_0$, we can choose $\theta_1' \in (\theta_1 + \Gamma_0) \cap G_0$ and $\epsilon_1', \epsilon_2' \in \Gamma_0 \setminus \{0\}$ such that $\ker\theta_1' \cap \ker\epsilon_1' \cap \ker\epsilon_2' = \{0\}$ and $\beta' \in G_1 = \mathbb{Z}\theta_1' + \mathbb{Z}\epsilon_1' + \mathbb{Z}\epsilon_2'$ (cf. (4.25)). Moreover, we note that $\{x^{\pm\alpha}\partial \mid \alpha \in \{\theta_1', \epsilon_1', \epsilon_2'\},\ \partial \in \ker\alpha\}$ generates the simple generalized divergence-free Lie algebra $\mathcal{S}(G_1, D)$. As (4.28) and (4.32) show

$$x^{\pm\alpha}\partial.M(\rho_\lambda, \Gamma_1) = \{0\} \quad \text{for } \alpha \in \left\{\theta_1', \epsilon_1', \epsilon_2'\right\} \text{ and } \partial \in \ker\alpha, \tag{4.33}$$

we can deduce

$$x^\tau\partial.M(\rho_\lambda, \Gamma_1) = \{0\} \quad \text{for } \tau \in G_1 \setminus \{0\} \text{ and } \partial \in \ker\tau. \tag{4.34}$$

In particular,

$$x^{\beta'}\partial.M(\rho_\lambda, \Gamma_1) = \{0\} \quad \text{for } \partial \in \ker\beta'. \tag{4.35}$$

Since $\beta' \in \Gamma_1 \setminus \Gamma_0$ is arbitrary, (4.32) and (4.35) yield

$$M(\rho_\lambda, \Gamma_1) \simeq \bigoplus_{\nu \in \Gamma_1} \mathbb{F}w_\nu, \quad \text{where each component is a trivial submodule of } \mathcal{S}(\Gamma_1, D). \tag{4.36}$$

Case 2. There exists some $k_0 \in K$ such that

$$M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \mathscr{A}_{\eta'_{k_0}}\big(\mathcal{S}(\Gamma_0, D)\big) \quad \text{for some } \eta'_{k_0} \in D^*\backslash\{0\}. \tag{4.37}$$

To begin with, we want to confirm which cases the submodules in (C3) are isomorphic to. Write $M_\theta = \mathbb{F}w'_\theta$ for $\theta \in \Gamma$.

Since $M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \mathscr{A}_{\eta'_{k_0}}(\mathcal{S}(\Gamma_0, D))$, (4.16) shows

$$x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.w'_{\rho_\lambda+k_0\theta_1+\beta+l\epsilon_1+m\epsilon_2} = \big(\beta + l\epsilon_1 + m\epsilon_2 + \mu'_{k_0}\big)^2(\partial)w'_{\rho_\lambda+k_0\theta_1+\beta+l\epsilon_1+m\epsilon_2} \tag{4.38}$$

for $\beta \in \Gamma_0$ and $l, m \in \mathbb{Z}$ such that $\beta + l\epsilon_1 + m\epsilon_2 + \mu'_{k_0} \neq 0 \neq \beta + l\epsilon_1 + m\epsilon_2 + \mu'_{k_0} + \epsilon_s$, where $\partial \in \ker\epsilon_s$ and $s \in \{1, 2\}$. Pick $\partial' \in \ker\epsilon_1 \backslash \ker\epsilon_2$. Fix any $\beta_0 \in \Gamma_0$. From (4.38) we see that there exist $l', m' \in \mathbb{Z}$ such that

$$x^{-\epsilon_1}\partial'.x^{\epsilon_1}\partial'.w'_{\rho_\lambda+k_0\theta_1+\beta_0+l'\epsilon_1+m'\epsilon_2} \neq 0, \tag{4.39}$$

which then implies, in (C3),

$$M(\rho_\lambda + \beta_0, G_0) \text{ can only be isomorphic to one of the first four cases.} \tag{4.40}$$

So by (4.23) and (4.24) we have

$$\begin{aligned} &x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.w'_{\rho_\lambda+\beta_0+k_0\theta_1+l\epsilon_1+m\epsilon_2} \\ &\quad = \big(k_0\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_{\beta_0}\big)^2(\partial)w'_{\rho_\lambda+\beta_0+k_0\theta_1+l'\epsilon_1+m'\epsilon_2} \end{aligned} \tag{4.41}$$

for $l, m \in \mathbb{Z}$ such that $k_0\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_{\beta_0} \neq 0 \neq k_0\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_{\beta_0} + \epsilon_s$, where $\partial \in \ker\epsilon_s$ and $s \in \{1, 2\}$. Comparing (4.41) with (4.38), we see that

$$\big(k_0\theta_1 + l_s\epsilon_1 + m_s\epsilon_2 + \mu''_{\beta_0}\big)^2(\partial) = \big(\beta_0 + l_s\epsilon_1 + m_s\epsilon_2 + \mu'_{k_0}\big)^2(\partial) \tag{4.42}$$

for $l_s, m_s \in \mathbb{Z}$ such that $\beta_0 + l_s\epsilon_1 + m_s\epsilon_2 + \mu'_{k_0} \neq 0 \neq \beta_0 + l_s\epsilon_1 + m_s\epsilon_2 + \mu'_{k_0} + \epsilon_s$ and $k_0\theta_1 + l_s\epsilon_1 + m_s\epsilon_2 + \mu''_{\beta_0} \neq 0 \neq k_0\theta_1 + l_s\epsilon_1 + m_s\epsilon_2 + \mu''_{\beta_0} + \epsilon_s$, where $\partial \in \ker\epsilon_s$ and $s \in \{1, 2\}$. Moreover, since there exist infinite pairs of such l_s and m_s for either $s \in \{1, 2\}$, we can deduce from (4.42) that

$$\big(k_0\theta_1 + \mu''_{\beta_0}\big)(\partial) = \big(\beta_0 + \mu'_{k_0}\big)(\partial) \quad \text{for } \partial \in (\ker\epsilon_1 \cup \ker\epsilon_2)\backslash(\ker\epsilon_1 \cap \ker\epsilon_2), \tag{4.43}$$

$$\big(k_0\theta_1 + \mu''_{\beta_0}\big)^2\big(\partial''\big) = \big(\beta_0 + \mu'_{k_0}\big)^2\big(\partial''\big) \quad \text{for } \partial'' \in \ker\epsilon_1 \cap \ker\epsilon_2. \tag{4.44}$$

Suppose that $(k_0\theta_1 + \mu''_{\beta_0})(\partial'') \neq (\beta_0 + \mu'_{k_0})(\partial'')$ for some $\partial'' \in \ker\epsilon_1 \cap \ker\epsilon_2$. Then (4.44) gives rise to

$$\big(k_0\theta_1 + \mu''_{\beta_0}\big)\big(\partial''\big) = -\big(\beta_0 + \mu'_{k_0}\big)\big(\partial''\big) \neq 0. \tag{4.45}$$

We will see this leads to a contradiction. Recall that $\partial' \in \ker\epsilon_1 \backslash \ker\epsilon_2$. Choose $l_1, m_1 \in \mathbb{Z}$ such that $\beta_0 + l_1\epsilon_1 + m_1\epsilon_2 + \mu'_{k_0} \neq 0 \neq \beta_0 + (l_1 + 1)\epsilon_1 + m_1\epsilon_2 + \mu'_{k_0}$, $k_0\theta_1 + l_1\epsilon_1 + m_1\epsilon_2 + \mu''_{\beta_0} \neq 0 \neq k_0\theta_1 + (l_1 + 1)\epsilon_1 + m_1\epsilon_2 + \mu''_{\beta_0}$ and

$$\big(\beta_0 + l_1\epsilon_1 + m_1\epsilon_2 + \mu'_{k_0}\big)\big(\partial'\big) \neq 0. \tag{4.46}$$

Observe that (4.37) and (4.40) enable us to choose nonzero vectors $u_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}$ and $u'_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}$ in $M_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}$, respectively, such that,

$$\begin{aligned} & x^{\epsilon_1}\partial.w'_{\rho_\lambda+k_0\theta_1+\beta_0+l_1\epsilon_1+m_1\epsilon_2} \\ &\quad = \left(\beta_0+l_1\epsilon_1+m_1\epsilon_2+\mu'_{k_0}\right)(\partial)u_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}, \end{aligned} \tag{4.47}$$

$$\begin{aligned} & x^{\epsilon_1}\partial.w'_{\rho_\lambda+k_0\theta_1+\beta_0+l_1\epsilon_1+m_1\epsilon_2} \\ &\quad = \left(k_0\theta_1+l_1\epsilon_1+m_1\epsilon_2+\mu''_{\beta_0}\right)(\partial)u'_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2} \end{aligned} \tag{4.48}$$

for $\partial\in\ker\epsilon_1$. Since (4.43) and (4.46) indicate

$$\left(k_0\theta_1+l_1\epsilon_1+m_1\epsilon_2+\mu''_{\beta_0}\right)\left(\partial'\right)=\left(\beta_0+l_1\epsilon_1+m_1\epsilon_2+\mu'_{k_0}\right)\left(\partial'\right)\neq 0, \tag{4.49}$$

taking $\partial=\partial'$ in (4.47) and (4.48), we get

$$u_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}=u'_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}. \tag{4.50}$$

On the other hand, (4.45) shows

$$\left(k_0\theta_1+l_1\epsilon_1+m_1\epsilon_2+\mu''_{\beta_0}\right)\left(\partial''\right)=-\left(\beta_0+l_1\epsilon_1+m_1\epsilon_2+\mu'_{k_0}\right)\left(\partial''\right)\neq 0. \tag{4.51}$$

Taking $\partial=\partial''$ in (4.47) and (4.48), we get

$$u_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}=-u'_{\rho_\lambda+k_0\theta_1+\beta_0+(l_1+1)\epsilon_1+m_1\epsilon_2}, \tag{4.52}$$

which contradicts (4.50). So we must have

$$\left(k_0\theta_1+\mu''_{\beta_0}\right)\left(\partial''\right)=\left(\beta_0+\mu'_{k_0}\right)\left(\partial''\right)\quad\text{for } \partial''\in\ker\epsilon_1\cap\ker\epsilon_2. \tag{4.53}$$

As $\ker\epsilon_1+\ker\epsilon_2=D$, (4.43) and (4.53) lead to

$$\mu''_{\beta_0}-\beta_0=\mu'_{k_0}-k_0\theta_1\quad\text{for any } \beta_0\in\Gamma_0. \tag{4.54}$$

Now we can confirm which cases the submodules in (C3) are isomorphic to. Since $-\mu'_{k_0}\in\Gamma_0$, we deduce from (4.54) that

$$\mu''_{-\mu'_{k_0}}=\mu'_{k_0}-k_0\theta_1-\mu'_{k_0}=-k_0\theta_1\in G_0. \tag{4.55}$$

Moreover, the fact $M(\rho_\lambda+k_0\theta_1,\Gamma_0)\simeq\mathscr{A}_{\eta'_{k_0}}(\mathcal{S}(\Gamma_0,D))$ and Eq. (4.16) give

$$x^{\epsilon_s}\partial.M_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}-\epsilon_s}=\{0\}\quad\text{for all } \partial\in\ker\epsilon_s \text{ and } s\in\{1,2\}, \tag{4.56}$$

$$x^{\epsilon_s}\partial.M_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}}\neq\{0\}\quad\text{for some } s\in\{1,2\} \text{ and } \partial\in\ker\epsilon_s. \tag{4.57}$$

So (C3), (4.40), (4.55)–(4.57) and $\ker\epsilon_1+\ker\epsilon_2=D$ show

$$M\left(\rho_\lambda-\mu'_{k_0},G_0\right)\simeq\mathscr{A}_\eta\left(\mathcal{S}(G_0,D)\right)\quad\text{for some } \eta\in D^*\backslash\{0\}. \tag{4.58}$$

For any $\beta \in \Gamma_0$, if $\mu''_\beta \in G_0$, we have

$$\beta = -\mu'_{k_0} + \mu''_\beta + k_0\theta_1 \in -\mu'_{k_0} + G_0 \tag{4.59}$$

by (4.54). Conversely, $\beta \in \Gamma_0 \backslash (-\mu'_{k_0} + G_0)$ implies $\mu''_\beta \notin G_0$. Thus this together with (C3), (4.40) and (4.58) imply

$$M(\rho_\lambda + \beta, G_0) = M\big(\rho_\lambda - \mu'_{k_0}, G_0\big) \simeq \mathscr{A}_\eta\big(\mathcal{S}(G_0, D)\big) \tag{4.60}$$

for $\beta \in \Gamma_0 \cap (-\mu'_{k_0} + G_0)$, and

$$\begin{aligned} &M(\rho_\lambda + \beta, G_0) \simeq \mathscr{M}_{\mu''_\beta}\big(\mathcal{S}(G_0, D)\big) = \mathscr{M}_{\beta + \mu'_{k_0} - k_0\theta_1}\big(\mathcal{S}(G_0, D)\big) \\ &\quad \text{for } \beta \in \Gamma_0 \backslash \big(-\mu'_{k_0} + G_0\big). \end{aligned} \tag{4.61}$$

Then we want to confirm which cases the submodules in (C2) are isomorphic to.

Fix $\beta \in \Gamma_0$ and $k \in \mathbb{Z}$. Note that (4.23), (4.24), (4.60) and (4.61) give

$$x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.w'_{\rho_\lambda + k\theta_1 + \beta + l\epsilon_1 + m\epsilon_2} = \big(k\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_\beta\big)^2(\partial)\, w'_{\rho_\lambda + k\theta_1 + \beta + l\epsilon_1 + m\epsilon_2} \tag{4.62}$$

for $l, m \in \mathbb{Z}$ such that $k\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_\beta \neq 0 \neq k\theta_1 + l\epsilon_1 + m\epsilon_2 + \mu''_\beta + \epsilon_s$, where $\partial \in \ker \epsilon_s$ and $s \in \{1, 2\}$. Pick $\partial' \in \ker \epsilon_1 \backslash \ker \epsilon_2$. Then (4.62) shows that there exist $l', m' \in \mathbb{Z}$ such that

$$x^{-\epsilon_1}\partial'.x^{\epsilon_1}\partial'.w'_{\rho_\lambda + k\theta_1 + \beta + l'\epsilon_1 + m'\epsilon_2} \neq 0, \tag{4.63}$$

which further implies, in (C2),

$$M(\rho_\lambda + k\theta_1, \Gamma_0) \text{ can only be isomorphic to one of the first four cases.} \tag{4.64}$$

So (4.15) and (4.16) yield

$$x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.w'_{\rho_\lambda + k\theta_1 + \beta + l\epsilon_1 + m\epsilon_2} = \big(\beta + l\epsilon_1 + m\epsilon_2 + \mu'_k\big)^2(\partial)\, w'_{\rho_\lambda + k\theta_1 + \beta + l\epsilon_1 + m\epsilon_2} \tag{4.65}$$

for $l, m \in \mathbb{Z}$ such that $\beta + l\epsilon_1 + m\epsilon_2 + \mu'_k \neq 0 \neq \beta + l\epsilon_1 + m\epsilon_2 + \mu'_k + \epsilon_s$, where $\partial \in \ker \epsilon_s$ and $s \in \{1, 2\}$. By the similar arguments as those from (4.38) to (4.54), we get

$$\mu'_k - k\theta_1 = \mu''_\beta - \beta \quad \text{for any } k \in \mathbb{Z}. \tag{4.66}$$

So combining (4.54) with (4.66), we obtain

$$\mu'_k - k\theta_1 = \mu''_\beta - \beta = \mu'_{k_0} - k_0\theta_1 \quad \text{for any } k \in \mathbb{Z},\ \beta \in \Gamma_0. \tag{4.67}$$

Since $\mu'_{k_0} \in \Gamma_0$, from (4.9) and (4.67), we deduce

$$\mu'_k = (k - k_0)\theta_1 + \mu'_{k_0} \notin \Gamma_0 \quad \text{for } k \in K \backslash \{k_0\}. \tag{4.68}$$

Thus (4.64) and (C2) imply

$$M(\rho_\lambda+k\theta_1,\Gamma_0)\simeq\mathscr{M}_{\mu'_k}\big(\mathcal{S}(\Gamma_0,D)\big)=\mathscr{M}_{\mu'_{k_0}+(k-k_0)\theta_1}\big(\mathcal{S}(\Gamma_0,D)\big)\quad\text{for } k\in K\backslash\{k_0\}.\tag{4.69}$$

Next we sew the $\mathcal{S}(\Gamma_0,D)$-submodules $M(\rho_\lambda+k\theta_1,\Gamma_0)$ for $k\in K$ together, and derive how $\mathcal{S}(\Gamma_1,D)$ acts on $M(\rho_\lambda,\Gamma_1)$. We fix some $\beta_0\in\Gamma_0\backslash(-\mu'_{k_0}+G_0)$ in this part.

For any $k\in K\backslash\{k_0\}$, (4.15) and (4.69) enable us to choose $\{0\neq v'_{\rho_\lambda+k\theta_1+\beta}\in M_{\rho_\lambda+k\theta_1+\beta}\mid\beta\in\Gamma_0\}$ such that

$$x^\alpha\partial.v'_{\rho_\lambda+k\theta_1+\beta}=\big(\beta+\mu'_{k_0}+(k-k_0)\theta_1\big)(\partial)v'_{\rho_\lambda+k\theta_1+\beta+\alpha}\tag{4.70}$$

for $\alpha\in\Gamma_0\backslash\{0\}$ and $\partial\in\ker\alpha$. Moreover, by (4.16) and (4.37), we can choose $\{0\neq v'_{\rho_\lambda+k_0\theta_1+\beta}\in M_{\rho_\lambda+k_0\theta_1+\beta}\mid\beta\in\Gamma_0\}$ such that

$$x^\alpha\partial.v'_{\rho_\lambda+k_0\theta_1+\beta}=\big(\beta+\mu'_{k_0}\big)(\partial)v'_{\rho_\lambda+k_0\theta_1+\beta+\alpha}\tag{4.71}$$

for $\partial\in\ker\alpha$, $\alpha\in\Gamma_0\backslash\{0\}$ and $\beta\in\Gamma_0\backslash\{-\mu'_{k_0}\}$, and

$$x^\alpha\partial.v'_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}}=\eta'_{k_0}(\partial)v'_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}+\alpha}\quad\text{for }\partial\in\ker\alpha\text{ and }\alpha\in\Gamma_0\backslash\{0\}.\tag{4.72}$$

On the other hand, since $\beta_0\in\Gamma_0\backslash(-\mu'_{k_0}+G_0)$, (4.23) and (4.61) enable us to choose $\{0\neq v''_{\rho_\lambda+\beta_0+\nu}\in M_{\rho_\lambda+\beta_0+\nu}\mid\nu\in G_0\}$ such that

$$x^\alpha\partial.v''_{\rho_\lambda+\beta_0+\nu}=\big(\nu+\mu'_{k_0}+\beta_0-k_0\theta_1\big)(\partial)v''_{\rho_\lambda+\beta_0+\nu+\alpha}\tag{4.73}$$

for $\alpha\in G_0\backslash\{0\}$ and $\partial\in\ker\alpha$. Assume

$$v''_{\rho_\lambda+\beta_0+k\theta_1+\gamma}=\hat{b}_{k,\gamma}v'_{\rho_\lambda+\beta_0+k\theta_1+\gamma}\quad\text{for }k\in K\text{ and }\gamma\in\Gamma_0\cap G_0,\tag{4.74}$$

where $\hat{b}_{k,\gamma}\in\mathbb{F}$.

We now show that $\hat{b}_{k,\gamma}=\hat{b}_k$ is independent of γ. Since $\beta_0\in\Gamma_0\backslash(-\mu'_{k_0}+G_0)$, we have

$$\beta_0+\gamma\neq-\mu'_{k_0}\quad\text{for any }\gamma\in\Gamma_0\cap G_0.\tag{4.75}$$

So fixing any $k\in K$, from (4.70) or (4.71), and from (4.73) we respectively derive

$$x^\alpha\partial.v'_{\rho_\lambda+k\theta_1+\beta_0+\gamma}=\big(\beta_0+\gamma+\mu'_k\big)(\partial)v'_{\rho_\lambda+k\theta_1+\beta_0+\gamma+\alpha},\tag{4.76}$$

$$x^\alpha\partial.v''_{\rho_\lambda+\beta_0+k\theta_1+\gamma}=\big(\beta_0+\gamma+\mu'_k\big)(\partial)v''_{\rho_\lambda+\beta_0+k\theta_1+\gamma+\alpha}\tag{4.77}$$

for $\alpha\in(\Gamma_0\cap G_0)\backslash\{0\}$, $\partial\in\ker\alpha$ and $\gamma\in\Gamma_0\cap G_0$, where $\mu'_k=\mu'_{k_0}+(k-k_0)\theta_1$. From $\beta_0\in\Gamma_0\backslash(-\mu'_{k_0}+G_0)$ again, it follows

$$\beta_0+\mu'_k+\gamma=\beta_0+\mu'_{k_0}+(k-k_0)\theta_1+\gamma\neq0\quad\text{for any }\gamma\in\Gamma_0\cap G_0.\tag{4.78}$$

We see that $\hat{b}_{k,\gamma}=\hat{b}_{k,0}$ in two cases. On one hand, for $\alpha'\in(\Gamma_0\cap G_0)\backslash\{0\}$ such that $\ker\alpha'\neq$

$\ker(\beta_0+\mu'_k)$, taking $\alpha=\alpha'$, $\gamma=0$ and $\partial\in\ker\alpha'\setminus\ker(\beta_0+\mu'_k)$ in (4.76) and (4.77), we obtain

$$\hat{b}_{k,0}=\hat{b}_{k,\alpha'}. \tag{4.79}$$

Noticing that such α' does exist (e.g. ϵ_1 or ϵ_2 does the trick), we denote one by α'_0. On the other hand, for $\alpha''\in(\Gamma_0\cap G_0)\setminus\{0\}$ such that $\ker\alpha''=\ker(\beta_0+\mu'_k)$, we first have

$$\ker\big(\alpha''-\alpha'_0\big)\neq\ker\big(\beta_0+\mu'_k\big), \tag{4.80}$$

which implies $\hat{b}_{k,0}=\hat{b}_{k,\alpha''-\alpha'_0}$ with α' replaced by $\alpha''-\alpha'_0$ in (4.79). Moreover, it follows from (4.78) that

$$\ker\alpha''=\ker\big(\beta_0+\mu'_k\big)=\ker\big(\beta_0+\mu'_k+\alpha''\big), \tag{4.81}$$

which implies $\ker\alpha'_0\neq\ker(\beta_0+\mu'_k+\alpha''-\alpha'_0)$. Thus, putting $\alpha=\alpha'_0$, $\gamma=\alpha''-\alpha'_0$ and taking $\partial\in\ker\alpha'_0\setminus\ker(\beta_0+\mu'_k+\alpha''-\alpha'_0)$ in (4.76) and (4.77), we obtain

$$\hat{b}_{k,\alpha''}=\hat{b}_{k,\alpha''-\alpha'_0}=\hat{b}_{k,0}. \tag{4.82}$$

So combining (4.79) with (4.82), we get

$$\hat{b}_{k,\gamma}=\hat{b}_{k,0}=\hat{b}_k\quad\text{for }\gamma\in\Gamma_0\cap G_0. \tag{4.83}$$

Recall that $\{k\theta_1\mid k\in K\}$ is the set of all representatives of cosets of Γ_0 in Γ_1 (cf. (4.9)). Multiplying $\{v'_{\rho_\lambda+k\theta_1+\beta}\mid\beta\in\Gamma_0\}$ by $\hat{b}_k$ for each $k\in K$, we get a new set

$$\{0\neq v_{\rho_\lambda+\varrho}\in M_{\rho_\lambda+\varrho}\mid\varrho\in\Gamma_1\}. \tag{4.84}$$

We then consider the action of $\mathcal{S}(\Gamma_1,D)$ on $M(\rho_\lambda,\Gamma_1)$.

First, (4.70), (4.71) and (4.72) show

$$x^\alpha\partial.v_{\rho_\lambda+\varrho}=\big(\varrho+\mu'_{k_0}-k_0\theta_1\big)(\partial)v_{\rho_\lambda+\varrho+\alpha} \tag{4.85}$$

for $\alpha\in\Gamma_0\setminus\{0\}$, $\partial\in\ker\alpha$ and $\varrho\in\Gamma_1\setminus\{k_0\theta_1-\mu'_{k_0}\}$, and

$$x^\alpha\partial.v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}}=\eta'_{k_0}(\partial)v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}+\alpha}\quad\text{for }\alpha\in\Gamma_0\setminus\{0\},\ \partial\in\ker\alpha, \tag{4.86}$$

while (4.73) and (4.74) imply

$$x^\tau\partial'.v_{\rho_\lambda+\beta_0+\nu}=\big(\nu+\beta_0+\mu'_{k_0}-k_0\theta_1\big)\big(\partial'\big)v_{\rho_\lambda+\beta_0+\nu+\tau} \tag{4.87}$$

for $\tau\in G_0\setminus\{0\}$, $\partial'\in\ker\tau$ and $\nu\in G_0$.

Second, we determine the action of $\mathcal{S}(G_0,D)$ on $M(\rho_\lambda,\Gamma_1)$. Since $\mathcal{X}_0=\{x^{\pm\alpha}\partial\mid\alpha\in\{\theta_1,\epsilon_1,\epsilon_2\},\ \partial\in\ker\alpha\}$ generates $\mathcal{S}(G_0,D)$, we only need to determine the action of $\mathcal{X}_0$ on $M(\rho_\lambda,\Gamma_1)$. Recall that $\epsilon_1,\epsilon_2\in\Gamma_0$. Then (4.85) and (4.86) show

$$x^{\pm\epsilon_s}\partial.v_{\rho_\lambda+\varrho}=\big(\varrho+\mu'_{k_0}-k_0\theta_1\big)(\partial)v_{\rho_\lambda+\varrho\pm\epsilon_s} \tag{4.88}$$

for $\varrho \in \Gamma_1 \backslash \{k_0\theta_1 - \mu'_{k_0}\}$, $\partial \in \ker \epsilon_s$ and $s \in \{1, 2\}$, and

$$x^{\pm\epsilon_s}\partial.v_{\rho_\lambda + k_0\theta_1 - \mu'_{k_0}} = \eta'_{k_0}(\partial) v_{\rho_\lambda + k_0\theta_1 - \mu'_{k_0} \pm \epsilon_s} \quad \text{for } \partial \in \ker \epsilon_s \text{ and } s \in \{1, 2\}. \tag{4.89}$$

Fix any $\varrho \in \Gamma_1 \backslash \{k_0\theta_1 - \mu'_{k_0}\}$. As $\Gamma_1 = \Gamma_0 + \mathbb{Z}\theta_1 = \Gamma_0 + G_0$, we see that

$$\varrho = \beta + \nu \quad \text{for some } \beta \in \Gamma_0 \text{ and } \nu \in G_0. \tag{4.90}$$

We then derive how $x^{\pm\theta_1}\partial$ acts on $v_{\rho_\lambda + \varrho}$ in two cases. If $\ker \theta_1 = \ker(\varrho + \mu'_{k_0} - k_0\theta_1)$, (4.60) or (4.61) gives

$$x^{\pm\theta_1}\partial.v_{\rho_\lambda + \varrho} = 0 = \big(\varrho + \mu'_{k_0} - k_0\theta_1\big)(\partial) v_{\rho_\lambda + \varrho \pm \theta_1} \quad \text{for } \partial \in \ker \theta_1. \tag{4.91}$$

Otherwise, we have $\dim(\ker \theta_1 \cap \ker(\varrho + \mu'_{k_0} - k_0\theta_1)) = 1$. Recall that $\varrho = \beta + \nu$ (cf. (4.90)). Choose $l, m \in \mathbb{Z}$ such that

$$\ker \theta_1 \cap \ker\big(\varrho + \mu'_{k_0} - k_0\theta_1\big) \cap \ker(\beta - \beta_0 + l\epsilon_1 + m\epsilon_2) = \{0\}. \tag{4.92}$$

Pick nonzero $\partial_1 \in \ker \theta_1 \cap \ker(\varrho + \mu'_{k_0} - k_0\theta_1)$ and $\partial_2 \in \ker \theta_1 \cap \ker(\beta - \beta_0 + l\epsilon_1 + m\epsilon_2)$, then $\{\partial_1, \partial_2\}$ forms a basis of $\ker \theta_1$ and $(\varrho + \mu'_{k_0} - k_0\theta_1)(\partial_2) \neq 0$. On one hand, (4.60) or (4.61) shows

$$x^{\pm\theta_1}\partial_1.v_{\rho_\lambda + \varrho} = 0 = \big(\varrho + \mu'_{k_0} - k_0\theta_1\big)(\partial_1) v_{\rho_\lambda + \varrho \pm \theta_1}. \tag{4.93}$$

On the other hand, since (4.87) gives the action of $x^{\pm\theta_1}\partial_2$ on $M(\rho_\lambda + \beta_0, G_0)$, we extend it to the action of $x^{\pm\theta_1}\partial_2$ on $v_{\rho_\lambda + \varrho}$ with the assistance of (4.85). Notice that $\beta - \beta_0 + l\epsilon_1 + m\epsilon_2 \in \Gamma_0 \backslash \{0\}$ and $M_{\rho_\lambda + \beta_0 - l\epsilon_1 - m\epsilon_2 + \nu} \subseteq M(\rho_\lambda + \beta_0, G_0)$ (cf. (4.90)). The following diagram illustrates the idea:

$$\begin{tikzcd}
M_{\rho_\lambda + \varrho} \arrow[r, "x^{\pm\theta_1}\partial_2"] & M_{\rho_\lambda + \varrho \pm \theta_1} \\
M_{\rho_\lambda + \beta_0 - l\epsilon_1 - m\epsilon_2 + \nu} \arrow[u, "x^{\beta - \beta_0 + l\epsilon_1 + m\epsilon_2}\partial_2"] \arrow[r, "x^{\pm\theta_1}\partial_2"] & M_{\rho_\lambda + \beta_0 - l\epsilon_1 - m\epsilon_2 + \nu \pm \theta_1} \arrow[u, "x^{\beta - \beta_0 + l\epsilon_1 + m\epsilon_2}\partial_2"']
\end{tikzcd}$$

Since $\partial_2 \in \ker(\beta - \beta_0 + l\epsilon_1 + m\epsilon_2)$, we have

$$\big(\beta_0 - l\epsilon_1 - m\epsilon_2 + \nu + \mu'_{k_0} - k_0\theta_1\big)(\partial_2) = \big(\varrho + \mu'_{k_0} - k_0\theta_1\big)(\partial_2) \neq 0. \tag{4.94}$$

Applying (4.85) and (4.87) to

$$\begin{aligned} & x^{\pm\theta_1}\partial_2.x^{\beta - \beta_0 + l\epsilon_1 + m\epsilon_2}\partial_2.v_{\rho_\lambda + \beta_0 - l\epsilon_1 - m\epsilon_2 + \nu} \\ & \quad = x^{\beta - \beta_0 + l\epsilon_1 + m\epsilon_2}\partial_2.x^{\pm\theta_1}\partial_2.v_{\rho_\lambda + \beta_0 - l\epsilon_1 - m\epsilon_2 + \nu} \end{aligned} \tag{4.95}$$

and making use of (4.94), we get

$$x^{\pm\theta_1}\partial_2.v_{\rho_\lambda+\varrho} = \left(\varrho + \mu'_{k_0} - k_0\theta_1\right)(\partial_2)v_{\rho_\lambda+\varrho\pm\theta_1}. \tag{4.96}$$

So (4.91), (4.93) and (4.96) show

$$x^{\pm\theta_1}\partial.v_{\rho_\lambda+\varrho} = \left(\varrho + \mu'_{k_0} - k_0\theta_1\right)(\partial)v_{\rho_\lambda+\varrho\pm\theta_1}$$
$$\text{for } \varrho \in \Gamma_1\backslash\left\{k_0\theta_1 - \mu'_{k_0}\right\},\ \partial \in \ker\theta_1. \tag{4.97}$$

Afterwards we need to derive the action of $x^{\pm\theta_1}\partial$ on $v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}}$. Recall that $\ker\theta_1 \cap \ker\epsilon_1 \cap \ker\epsilon_2 = \{0\}$ (cf. (4.17)). Pick nonzero $\partial'_1 \in \ker\theta_1 \cap \ker\epsilon_1$ and $\partial'_2 \in \ker\theta_1 \cap \ker\epsilon_2$, then $\{\partial'_1, \partial'_2\}$ forms a basis of $\ker\theta_1$. Note that $\epsilon_2(\partial'_1) \neq 0$ and $\epsilon_1(\partial'_2) \neq 0$. Moreover, we pick $\tilde{\partial}_1 \in \ker\epsilon_1\backslash\ker\theta_1$ and $\tilde{\partial}_2 \in \ker\epsilon_2\backslash\ker\theta_1$. Then using (4.88), (4.89) and (4.97), we get

$$\begin{aligned} x^{\pm\theta_1}\partial'_1.v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}} &= \pm\frac{1}{\theta_1(\tilde{\partial}_1)\epsilon_1(\partial'_2)}\left[x^{-\epsilon_1}\tilde{\partial}_1, \left[x^{\pm\theta_1}\partial'_2, x^{\epsilon_1}\partial'_1\right]\right].v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}} \\ &= \eta'_{k_0}\left(\partial'_1\right)v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}\pm\theta_1}. \end{aligned} \tag{4.98}$$

Similarly, we can prove

$$x^{\pm\theta_1}\partial'_2.v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}} = \eta'_{k_0}\left(\partial'_2\right)v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}\pm\theta_1}. \tag{4.99}$$

Since $\mathcal{X}_0 = \{x^{\pm\alpha}\partial \mid \alpha \in \{\theta_1, \epsilon_1, \epsilon_2\},\ \partial \in \ker\alpha\}$ generates $\mathcal{S}(G_0, D)$, we can deduce from (4.88), (4.89), (4.97), (4.98) and (4.99) that

$$x^\tau\partial.v_{\rho_\lambda+\varrho} = \left(\varrho + \mu'_{k_0} - k_0\theta_1\right)(\partial)v_{\rho_\lambda+\varrho+\tau} \tag{4.100}$$

for $\varrho \in \Gamma_1\backslash\{k_0\theta_1 - \mu'_{k_0}\}$, $\tau \in G_0\backslash\{0\}$ and $\partial \in \ker\tau$, and

$$x^\tau\partial.v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}} = \eta'_{k_0}(\partial)v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}+\tau} \quad \text{for } \tau \in G_0\backslash\{0\} \text{ and } \partial \in \ker\tau. \tag{4.101}$$

Third, we determine the action of $x^{\beta'}\partial$ on $M(\rho_\lambda, \Gamma_1)$ for any $\beta' \in \Gamma_1\backslash\Gamma_0$ and $\partial \in \ker\beta'$. Fix any $\beta' \in \Gamma_1\backslash\Gamma_0$. Recall that we can choose $\theta'_1 \in (\theta_1 + \Gamma_0) \cap G_0$ and $\epsilon'_1, \epsilon'_2 \in \Gamma_0\backslash\{0\}$ such that $\ker\theta'_1 \cap \ker\epsilon'_1 \cap \ker\epsilon'_2 = \{0\}$ and $\beta' \in G_1 = \mathbb{Z}\theta'_1 + \mathbb{Z}\epsilon'_1 + \mathbb{Z}\epsilon'_2$ (cf. (4.25)). Observe that the set $\mathcal{X}_1 = \{x^{\pm\alpha}\partial \mid \alpha \in \{\theta'_1, \epsilon'_1, \epsilon'_2\},\ \partial \in \ker\alpha\}$ generates $\mathcal{S}(G_1, D)$, which is a subalgebra of $\mathcal{S}(\Gamma, D)$ and also a simple generalized divergence-free Lie algebra. As (4.85), (4.86), (4.100) and (4.101) give the action of $\mathcal{X}_1$ on $M(\rho_\lambda, \Gamma_1)$, we can derive

$$x^{\beta'}\partial.v_{\rho_\lambda+\varrho} = \left(\varrho + \mu'_{k_0} - k_0\theta_1\right)(\partial)v_{\rho_\lambda+\varrho+\beta'}$$
$$\text{for } \varrho \in \Gamma_1\backslash\left\{k_0\theta_1 - \mu'_{k_0}\right\} \text{ and } \partial \in \ker\beta', \tag{4.102}$$
$$x^{\beta'}\partial.v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}} = \eta'_{k_0}(\partial)v_{\rho_\lambda+k_0\theta_1-\mu'_{k_0}+\beta'} \quad \text{for } \partial \in \ker\beta'. \tag{4.103}$$

To sum up, (4.85), (4.86), (4.102) and (4.103) show

$$M(\rho_\lambda, \Gamma_1) \simeq \mathscr{A}_{\eta'_{k_0}}\left(\mathcal{S}(\Gamma_1, D)\right), \quad \text{where } \eta'_{k_0} \in D^*\backslash\{0\}. \tag{4.104}$$

Case 3. There exists some $k_0 \in K$ such that

$$M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \mathscr{M}_{\mu'_{k_0}}\big(\mathcal{S}(\Gamma_0, D)\big) \quad \text{for some } \mu'_{k_0} \in \Gamma_0. \tag{4.105}$$

In analogy with Case 2, we obtain

$$M(\rho_\lambda, \Gamma_1) \simeq \mathscr{M}_{\mu'_{k_0} - k_0\theta_1}\big(\mathcal{S}(\Gamma_1, D)\big), \quad \text{where } \mu'_{k_0} - k_0\theta_1 \in \Gamma_1. \tag{4.106}$$

Case 4. There exists some $k_0 \in K$ such that

$$M(\rho_\lambda + k_0\theta_1, \Gamma_0) \simeq \mathscr{B}_{\eta'_{k_0}}\big(\mathcal{S}(\Gamma_0, D)\big) \quad \text{for some } \eta'_{k_0} \in D^*\backslash\{0\}. \tag{4.107}$$

In analogy with Case 2, we obtain

$$M(\rho_\lambda, \Gamma_1) \simeq \mathscr{B}_{\eta'_{k_0}}\big(\mathcal{S}(\Gamma_1, D)\big), \quad \text{where } \eta'_{k_0} \in D^*\backslash\{0\}. \tag{4.108}$$

Case 5. For each $k \in K$,

$$M(\rho_\lambda + k\theta_1, \Gamma_0) \simeq \mathscr{M}_{\mu'_k}\big(\mathcal{S}(\Gamma_0, D)\big) \quad \text{with some } \mu'_k \in D^*\backslash\Gamma_0. \tag{4.109}$$

In analogy with Case 2, we get

$$\mu'_k - k\theta_1 = \mu'_0 \quad \text{for } k \in K, \tag{4.110}$$

and obtain

$$M(\rho_\lambda, \Gamma_1) \simeq \mathscr{M}_{\mu'_0}\big(\mathcal{S}(\Gamma_1, D)\big), \quad \text{where } \mu'_0 \in D^*\backslash\Gamma_1. \tag{4.111}$$

In summary, we get:
For any $\lambda \in J$, $M(\rho_\lambda, \Gamma_1)$ is isomorphic to

$$\text{(i)} \quad \mathscr{M}_{\bar\mu_\lambda}\big(\mathcal{S}(\Gamma_1, D)\big) \text{ for some } \bar\mu_\lambda \in D^*\backslash\Gamma_1; \tag{4.112}$$

$$\text{(ii)} \quad \mathscr{M}_{\bar\mu_\lambda}\big(\mathcal{S}(\Gamma_1, D)\big) \text{ for some } \bar\mu_\lambda \in \Gamma_1; \tag{4.113}$$

$$\text{(iii)} \quad \mathscr{A}_{\bar\eta_\lambda}\big(\mathcal{S}(\Gamma_1, D)\big) \text{ for some } \bar\eta_\lambda \in D^*\backslash\{0\}; \tag{4.114}$$

$$\text{(iv)} \quad \mathscr{B}_{\bar\eta_\lambda}\big(\mathcal{S}(\Gamma_1, D)\big) \text{ for some } \bar\eta_\lambda \in D^*\backslash\{0\}; \tag{4.115}$$

$$\text{(v)} \quad \bigoplus_{\nu \in \Gamma_1} \mathbb{F}w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(\Gamma_1, D). \tag{4.116}$$

Namely, Γ_1 satisfies condition (C1), which contradicts the maximality of Γ_0. So we must have $\Gamma = \Gamma_0$. Therefore, Lemma 4.1 follows from (C1), Lemma 2.1 and Theorem 2.2. □

5. The case $\dim D \geqslant 4$

In this section, we shall prove Theorem 2.3 under the condition $\dim D \geqslant 4$. We always assume $\dim D \geqslant 4$ throughout this section.

Let $M = \bigoplus_{\theta\in\Gamma} M_\theta$ be a Γ-graded $\mathcal{S}(\Gamma, D)$-module with $\dim M_\theta = 1$ for each $\theta \in \Gamma$. In order to prove Theorem 2.3 under the condition $\dim D \geqslant 4$, we first need to derive all the possible action of $\mathcal{S}(\Gamma, D)$ on M. Picking any three $\mathbb{F}$-linearly independent elements $\partial_1, \partial_2, \partial_3 \in D$, we set

$$D_0 = \mathbb{F}\partial_1 + \mathbb{F}\partial_2 + \mathbb{F}\partial_3. \tag{5.1}$$

To begin with, we have the following lemma.

Lemma 5.1. *The set* $\{x^\alpha\partial \mid \partial \in \ker\alpha \cap D_0,\ \alpha \in \Gamma$ *satisfying* $D_0 \not\subseteq \ker\alpha\}$ *acts on* M *in one of the following two ways*:

($P1$) $x^\alpha\partial.M = \{0\}$ *for* $\partial \in \ker\alpha \cap D_0$ *and* $\alpha \in \Gamma$ *satisfying* $D_0 \not\subseteq \ker\alpha$;
($P2$) *there exist* $\mu \in D^*$ *and* $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma$ *satisfying* $D_0 \not\subseteq \ker(\beta+\mu)\}$ *such that*

$$x^\alpha\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.2}$$

for $\partial \in \ker\alpha \cap D_0$, $\alpha \in \Gamma$ *satisfying* $D_0 \not\subseteq \ker\alpha$, *and* $\beta \in \Gamma$ *satisfying* $D_0 \not\subseteq \ker(\beta+\mu)$ *and* $D_0 \not\subseteq \ker(\beta+\mu+\alpha)$.

Proof. By Zorn's lemma and Lemma 4.1, there exists a subgroup Γ_1 of Γ which is maximal with the property:

(I) Γ_1 has a subgroup Γ_0 that satisfies $(\bigcap_{\alpha\in\Gamma_0}\ker\alpha) \cap D_0 = \{0\}$,
(II) $\{x^\alpha\partial \mid \partial \in \ker\alpha \cap D_0, \alpha \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha\}$ act on $\bigoplus_{\beta\in\Gamma_1} M_\beta$ in one of the following two ways:
($p1$) $x^\alpha\partial.M_\beta = \{0\}$ for $\beta \in \Gamma_1$, $\partial \in \ker\alpha \cap D_0$ and $\alpha \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha$;
($p2$) there exist $\mu \in D^*$ and $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker(\beta+\mu)\}$ such that

$$x^\alpha\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.3}$$

for $\partial \in \ker\alpha \cap D_0$, $\alpha \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha$, and $\beta \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker(\beta+\mu)$ and $D_0 \not\subseteq \ker(\beta+\mu+\alpha)$.

To prove the lemma, it suffices to show $\Gamma_1 = \Gamma$. Suppose that $\Gamma_1 \neq \Gamma$. We will see that this leads to a contradiction.

Choose $\theta_1 \in \Gamma\backslash\Gamma_1$ such that $D_0 \not\subseteq \ker\theta_1$. Let

$$\Gamma_2 = \Gamma_1 + \mathbb{Z}\theta_1. \tag{5.4}$$

We immediately see that Γ_2 satisfies (I). Moreover, we want to prove Γ_2 satisfies (II).

Suppose that Γ_1 satisfies ($p2$) in (II); the case in which Γ_1 satisfies ($p1$) can be proved similarly. Namely, there exist $\mu \in D^*$ and $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker(\beta+\mu)\}$ such that

$$x^\alpha\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.5}$$

for $\partial \in \ker\alpha \cap D_0$, $\alpha \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha$, and $\beta \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker(\beta+\mu)$ and $D_0 \not\subseteq$

$\ker(\beta+\mu+\alpha)$. We shall extend this to

$$\text{the action of } \{x^{\alpha}\partial \mid \partial \in \ker\alpha \cap D_0,\ \alpha \in \Gamma_2 \text{ satisfying } D_0 \nsubseteq \ker\alpha\} \text{ on } \bigoplus_{\beta\in\Gamma_2} M_\beta. \tag{5.6}$$

The fact that Γ_1 satisfies (I) enables us to choose $\epsilon_1, \epsilon_2 \in \Gamma_1$ such that

$$\ker\theta_1 \cap \ker\epsilon_1 \cap \ker\epsilon_2 \cap D_0 = \{0\}. \tag{5.7}$$

Set

$$G_0 = \mathbb{Z}\theta_1 + \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2. \tag{5.8}$$

Then $(\bigcap_{\nu\in G_0}\ker\nu) \cap D_0 = \{0\}$, and G_0 can be viewed as a subgroup of D_0^*. Hence we get a simple generalized divergence-free Lie algebra $\mathcal{S}(G_0, D_0)$, which is also a subalgebra of $\mathcal{S}(\Gamma, D)$.

We then proceed our proof in several steps.

Firstly, we extend the set $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma_1,\ D_0 \nsubseteq \ker(\beta+\mu)\}$ to $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma_2, D_0 \nsubseteq \ker(\beta+\mu)\}$, and determine the action of $\mathcal{S}(G_0, D_0)$ on $\bigoplus_{\beta\in\Gamma_2} M_\beta$.

Let

$$\{\beta_\varsigma \mid \varsigma \in \bar{I}\} \text{ be the set of all representatives of cosets of } \Gamma_1 \cap G_0 \text{ in } \Gamma_1. \tag{5.9}$$

Fix any $\varsigma \in \bar{I}$. Lemma 4.1 says that $M(\beta_\varsigma, G_0)$ is isomorphic to:

$$\text{(i)}\quad \mathscr{M}_{\mu_\varsigma}\big(\mathcal{S}(G_0, D_0)\big) \text{ for some } \mu_\varsigma \in D_0^*\backslash G_0; \tag{5.10}$$

$$\text{(ii)}\quad \mathscr{M}_{\mu_\varsigma}\big(\mathcal{S}(G_0, D_0)\big) \text{ for some } \mu_\varsigma \in G_0; \tag{5.11}$$

$$\text{(iii)}\quad \mathscr{A}_{\eta_\varsigma}\big(\mathcal{S}(G_0, D_0)\big) \text{ for some } \eta_\varsigma \in D_0^*\backslash\{0\}; \tag{5.12}$$

$$\text{(iv)}\quad \mathscr{B}_{\eta_\varsigma}\big(\mathcal{S}(G_0, D_0)\big) \text{ for some } \eta_\varsigma \in D_0^*\backslash\{0\}; \tag{5.13}$$

$$\text{(v)}\quad \bigoplus_{\nu\in G_0} \mathbb{F}w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(G_0, D_0). \tag{5.14}$$

If $M(\beta_\varsigma, G_0)$ is isomorphic to one of the first four cases (cf. (5.10)–(5.13)), then there exists some $\mu_\varsigma \in D_0^*$ so that, we can choose nonzero $v'_{\beta_\varsigma+\nu} \in M_{\beta_\varsigma+\nu}$ with $\nu \in G_0$ satisfying $\nu|_{D_0} + \mu_\varsigma \neq 0$ such that

$$x^{\alpha}\partial.v'_{\beta_\varsigma+\nu} = (\nu+\mu_\varsigma)(\partial)v'_{\beta_\varsigma+\nu+\alpha} \tag{5.15}$$

for $\alpha \in G_0\backslash\{0\}$, $\partial \in \ker\alpha \cap D_0$ and $\nu \in G_0$ satisfying $\nu|_{D_0} + \mu_\varsigma \neq 0 \neq \nu|_{D_0} + \mu_\varsigma + \alpha|_{D_0}$.

Recall that $\epsilon_1, \epsilon_2 \in \Gamma_1$ (cf. (5.7)). Pick $\partial' \in (\ker\epsilon_1 \cap D_0)\backslash(\ker\epsilon_2 \cap D_0)$. Then (5.5) indicates

$$x^{\epsilon_1}\partial'.v_{\beta_\varsigma+l'\epsilon_1+m'\epsilon_2} \neq 0 \quad \text{for some } l', m' \in \mathbb{Z}, \tag{5.16}$$

which implies,

$$M(\beta_\varsigma, G_0) \text{ can only be isomorphic to one of the first four cases } \big(\text{i.e. } (5.10)\text{–}(5.13)\big). \tag{5.17}$$

So (5.15) shows

$$x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.v_{\beta_\varsigma+l\epsilon_1+m\epsilon_2}=(l\epsilon_1+m\epsilon_2+\mu_\varsigma)^2(\partial)v_{\beta_\varsigma+l\epsilon_1+m\epsilon_2} \tag{5.18}$$

for $l,m\in\mathbb{Z}$ such that $l\epsilon_1|_{D_0}+m\epsilon_2|_{D_0}+\mu_\varsigma\neq 0\neq l\epsilon_1|_{D_0}+m\epsilon_2|_{D_0}+\mu_\varsigma+\epsilon_s|_{D_0}$, where $\partial\in\ker\epsilon_s\cap D_0$ and $s\in\{1,2\}$. On the other hand, (5.5) tells

$$x^{-\epsilon_s}\partial.x^{\epsilon_s}\partial.v_{\beta_\varsigma+l\epsilon_1+m\epsilon_2}=(\beta_\varsigma+l\epsilon_1+m\epsilon_2+\mu)^2(\partial)v_{\beta_\varsigma+l\epsilon_1+m\epsilon_2} \tag{5.19}$$

for $l,m\in\mathbb{Z}$ such that $D_0\nsubseteq\ker(\beta_\varsigma+l\epsilon_1+m\epsilon_2+\mu)$ and $D_0\nsubseteq\ker(\beta_\varsigma+l\epsilon_1+m\epsilon_2+\mu+\epsilon_s)$, where $\partial\in\ker\epsilon_s\cap D_0$ and $s\in\{1,2\}$. By the similar arguments as those from (4.42) to (4.54), we can prove

$$\mu_\varsigma=(\mu+\beta_\varsigma)|_{D_0}\quad\text{for any }\varsigma\in\bar{I}. \tag{5.20}$$

So this together with (5.15) and (5.17) enable us to choose nonzero $v'_{\beta_\varsigma+\nu}\in M_{\beta_\varsigma+\nu}$ with $\nu\in G_0$ satisfying $D_0\nsubseteq\ker(\nu+\mu+\beta_\varsigma)$ such that

$$x^\alpha\partial.v'_{\beta_\varsigma+\nu}=(\nu+\mu+\beta_\varsigma)(\partial)v'_{\beta_\varsigma+\nu+\alpha} \tag{5.21}$$

for $\partial\in\ker\alpha\cap D_0$, $\alpha\in G_0\backslash\{0\}$, and $\nu\in G_0$ satisfying $D_0\nsubseteq\ker(\nu+\mu+\beta_\varsigma)$ and $D_0\nsubseteq\ker(\nu+\mu+\beta_\varsigma+\alpha)$. Assume

$$v'_{\beta_\varsigma+\gamma}=\hat{c}_{\varsigma,\gamma}v_{\beta_\varsigma+\gamma}\quad\text{for }\gamma\in\Gamma_1\cap G_0\text{ such that }D_0\nsubseteq\ker(\beta_\varsigma+\gamma+\mu), \tag{5.22}$$

where $\hat{c}_{\varsigma,\gamma}\in\mathbb{F}$ (cf. (5.5)). By the similar arguments as those from (4.74) to (4.83), we can prove that

$$\hat{c}_{\varsigma,\gamma}=\hat{c}_\varsigma\text{ is independent of }\gamma. \tag{5.23}$$

Since $\Gamma_2=\Gamma_1+\mathbb{Z}\theta_1=\Gamma_1+G_0=\bigoplus_{\varsigma\in\bar{I}}(\beta_\varsigma+G_0)$, multiplying

$$\{v'_{\beta_\varsigma+\nu}\mid\nu\in G_0\text{ satisfying }D_0\nsubseteq\ker(\beta_\varsigma+\nu+\mu)\} \tag{5.24}$$

by $\frac{1}{\hat{c}_\varsigma}$ for each $\varsigma\in\bar{I}$, we get a set

$$\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma_2,\ D_0\nsubseteq\ker(\beta+\mu)\}, \tag{5.25}$$

which expands $\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma_1,\ D_0\nsubseteq\ker(\beta+\mu)\}$ by (5.22) and (5.23). Then (5.5) still holds and (5.21) shows

$$x^\alpha\partial.v_\beta=(\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.26}$$

for $\alpha\in G_0\backslash\{0\}$, $\partial\in\ker\alpha\cap D_0$ and $\beta\in\Gamma_2$ satisfying $D_0\nsubseteq\ker(\beta+\mu)$ and $D_0\nsubseteq\ker(\beta+\mu+\alpha)$.

Secondly, we determine the action of $\{x^\alpha\partial\mid\partial\in\ker\alpha\cap D_0,\ \alpha\in\Gamma_1$ such that $D_0\nsubseteq\ker\alpha$ and $\ker\alpha\cap D_0\neq\ker\theta_1\cap D_0\}$ on $\bigoplus_{\beta\in\Gamma_2}M_\beta$.

Fix any $\alpha'\in\Gamma_1$ such that $D_0\nsubseteq\ker\alpha'$ and $\ker\alpha'\cap D_0\neq\ker\theta_1\cap D_0$. Fix any $\varrho'\in\Gamma_2$ such that

$$D_0\nsubseteq\ker(\varrho'+\mu)\quad\text{and}\quad D_0\nsubseteq\ker(\varrho'+\mu+\alpha'). \tag{5.27}$$

We want to see how $x^{\alpha'}\partial$ acts on $v_{\varrho'}$ for $\partial\in\ker\alpha'\cap D_0$.

Since $\ker\theta_1 \cap \ker\alpha' \cap \ker\epsilon_1 \cap D_0 = \{0\}$ or $\ker\theta_1 \cap \ker\alpha' \cap \ker\epsilon_2 \cap D_0 = \{0\}$, without loss of generality, we assume that $\ker\theta_1 \cap \ker\alpha' \cap \ker\epsilon_1 \cap D_0 = \{0\}$. Let

$$G_1 = \mathbb{Z}\theta_1 + \mathbb{Z}\alpha' + \mathbb{Z}\epsilon_1. \tag{5.28}$$

Then $(\bigcap_{\nu\in G_1} \ker\nu) \cap D_0 = \{0\}$ and G_1 can be viewed as a subgroup of D_0^*. Hence we get a simple generalized divergence-free Lie algebra $\mathcal{S}(G_1, D_0)$, which is also a subalgebra of $\mathcal{S}(\Gamma, D)$. By Lemma 4.1, we know that $M(\varrho', G_1)$ is isomorphic to:

$$\text{(i)}\quad \mathscr{M}_{\mu'_{\varrho'}}\big(\mathcal{S}(G_1, D_0)\big) \text{ for some } \mu'_{\varrho'} \in D_0^*\backslash G_1; \tag{5.29}$$

$$\text{(ii)}\quad \mathscr{M}_{\mu'_{\varrho'}}\big(\mathcal{S}(G_1, D_0)\big) \text{ for some } \mu'_{\varrho'} \in G_1; \tag{5.30}$$

$$\text{(iii)}\quad \mathscr{A}_{\eta'_{\varrho'}}\big(\mathcal{S}(G_1, D_0)\big) \text{ for some } \eta'_{\varrho'} \in D_0^*\backslash\{0\}; \tag{5.31}$$

$$\text{(iv)}\quad \mathscr{B}_{\eta'_{\varrho'}}\big(\mathcal{S}(G_1, D_0)\big) \text{ for some } \eta'_{\varrho'} \in D_0^*\backslash\{0\}; \tag{5.32}$$

$$\text{(v)}\quad \bigoplus_{\nu\in G_1} \mathbb{F}w_\nu, \text{ where each component is a trivial submodule of } \mathcal{S}(G_1, D_0). \tag{5.33}$$

If $M(\varrho', G_1)$ is isomorphic to one of the first four cases (cf. (5.29)–(5.32)), then there exists some $\mu'_{\varrho'} \in D_0^*$ so that, we can choose nonzero $w'_{\varrho'+\nu} \in M_{\varrho'+\nu}$ with $\nu \in G_1$ satisfying $\nu|_{D_0} + \mu'_{\varrho'} \neq 0$ such that

$$x^\tau \partial.w'_{\varrho'+\nu} = \big(\nu + \mu'_{\varrho'}\big)(\partial) w'_{\varrho'+\nu+\tau} \tag{5.34}$$

for $\tau \in G_1\backslash\{0\}$, $\partial \in \ker\tau \cap D_0$ and $\nu \in G_1$ satisfying $\nu|_{D_0} + \mu'_{\varrho'} \neq 0 \neq \nu|_{D_0} + \mu'_{\varrho'} + \tau|_{D_0}$. Note that (5.26) gives the action of $x^\alpha\partial$ on $\bigoplus_{\beta\in\Gamma_2} M_\beta$ for $\partial \in \ker\alpha$ and $\alpha \in \{\pm\theta_1, \pm\epsilon_1\}$. By the similar arguments as those from (5.16) to (5.20), we can prove that

$$M\big(\varrho', G_1\big) \text{ is isomorphic to one of the first four cases } \big(\text{i.e. (5.29)–(5.32)}\big), \tag{5.35}$$

and

$$\mu'_{\varrho'} = \big(\mu + \varrho'\big)\big|_{D_0}. \tag{5.36}$$

Pick nonzero $\tilde\partial_1 \in \ker\theta_1 \cap \ker\alpha' \cap D_0$ and $\tilde\partial_2 \in \ker(\theta_1 + \epsilon_1) \cap \ker\alpha' \cap D_0$. Then $\{\tilde\partial_1, \tilde\partial_2\}$ forms a basis of $\ker\alpha' \cap D_0$. If $(\varrho' + \mu)(\tilde\partial_1) = 0$, then (5.27) and (5.34)–(5.36) indicate

$$x^{\alpha'}\tilde\partial_1.v_{\varrho'} = 0 = \big(\varrho' + \mu\big)(\tilde\partial_1) v_{\alpha'+\varrho'}. \tag{5.37}$$

Assume $(\varrho' + \mu)(\tilde\partial_1) \neq 0$. Since $\Gamma_2 = \Gamma_1 + \mathbb{Z}\theta_1$ and $\varrho' \in \Gamma_2$, we can write $\varrho' = \beta' + k'\theta_1$ for some $\beta' \in \Gamma_1$ and $k' \in \mathbb{Z}$. As $\tilde\partial_1 \in \ker\theta_1 \cap \ker\alpha' \cap D_0$, we have

$$\big(\beta' + \mu\big)(\tilde\partial_1) = \big(\varrho' + \mu\big)(\tilde\partial_1) \neq 0 \quad \text{and} \quad \big(\alpha' + \beta' + \mu\big)(\tilde\partial_1) = \big(\varrho' + \mu\big)(\tilde\partial_1) \neq 0; \tag{5.38}$$

namely, $D_0 \nsubseteq \ker(\beta' + \mu)$ and $D_0 \nsubseteq \ker(\beta' + \mu + \alpha')$. So by (5.5) and (5.26), we have

$$
\begin{aligned}
\big(\beta'+\mu\big)(\tilde{\partial}_1)x^{\alpha'}\tilde{\partial}_1.v_{\varrho'} &= x^{\alpha'}\tilde{\partial}_1.x^{k'\theta_1}\tilde{\partial}_1.v_{\beta'} = x^{k'\theta_1}\tilde{\partial}_1.x^{\alpha'}\tilde{\partial}_1.v_{\beta'} \\
&= \big(\beta'+\mu\big)^2(\tilde{\partial}_1)v_{\alpha'+\varrho'},
\end{aligned}
\tag{5.39}
$$

which implies

$$
x^{\alpha'}\tilde{\partial}_1.v_{\varrho'} = \big(\beta'+\mu\big)(\tilde{\partial}_1)v_{\alpha'+\varrho'} = \big(\varrho'+\mu\big)(\tilde{\partial}_1)v_{\alpha'+\varrho'}. \tag{5.40}
$$

Replacing $\tilde{\partial}_1$ by $\tilde{\partial}_2$ and replacing θ_1 by $\theta_1+\epsilon_1$ in (5.37)–(5.40), we can similarly prove

$$
x^{\alpha'}\tilde{\partial}_2.v_{\varrho'} = \big(\varrho'+\mu\big)(\tilde{\partial}_2)v_{\alpha'+\varrho'}. \tag{5.41}
$$

So we obtain

$$
x^{\alpha'}\partial.v_{\varrho'} = \big(\varrho'+\mu\big)(\partial)v_{\alpha'+\varrho'} \tag{5.42}
$$

for $\partial \in \ker\alpha' \cap D_0$, $\alpha' \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha'$ and $\ker\alpha' \cap D_0 \neq \ker\theta_1 \cap D_0$, and $\varrho' \in \Gamma_2$ satisfying $D_0 \not\subseteq \ker(\varrho'+\mu)$ and $D_0 \not\subseteq \ker(\varrho'+\mu+\alpha')$.

Thirdly, we determine the action of $\{x^\alpha\partial \mid \partial \in \ker\alpha \cap D_0,\ \alpha \in \Gamma_1$ such that $D_0 \not\subseteq \ker\alpha$ and $\ker\alpha \cap D_0 = \ker\theta_1 \cap D_0\}$ on $\bigoplus_{\beta\in\Gamma_2} M_\beta$.

Fix any $\alpha' \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha'$ and $\ker\alpha' \cap D_0 = \ker\theta_1 \cap D_0$. Fix any $\varrho' \in \Gamma_2$ such that

$$
D_0 \not\subseteq \ker\big(\varrho'+\mu\big) \quad \text{and} \quad D_0 \not\subseteq \ker\big(\varrho'+\mu+\alpha'\big). \tag{5.43}
$$

We want to see how $x^{\alpha'}\partial$ acts on $v_{\varrho'}$ for any $\partial \in \ker\alpha' \cap D_0$.

Since $\ker\alpha' \cap D_0 = \ker\theta_1 \cap D_0$, we have $\ker\alpha' \cap \ker\epsilon_1 \cap \ker\epsilon_2 \cap D_0 = \{0\}$. Let

$$
G_2 = \mathbb{Z}\alpha' + \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2. \tag{5.44}
$$

Then $(\bigcap_{\nu\in G_2}\ker\nu) \cap D_0 = \{0\}$ and G_2 can be viewed as a subgroup of D_0^*. Hence we get a simple generalized divergence-free Lie algebra $\mathcal{S}(G_2, D_0)$, which is also a subalgebra of $\mathcal{S}(\Gamma, D)$. Note that (5.26) gives the action of $x^{\pm\epsilon_s}\partial$ on $\bigoplus_{\beta\in\Gamma_2} M_\beta$ for $\partial \in \ker\epsilon_s$ and $s \in \{1,2\}$. By the similar arguments as those from (5.29) to (5.42), we can prove that

$$
x^{\alpha'}\partial.v_{\varrho'} = \big(\varrho'+\mu\big)(\partial)v_{\alpha'+\varrho'} \tag{5.45}
$$

for $\partial \in \ker\alpha' \cap D_0$, $\alpha' \in \Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha'$ and $\ker\alpha' \cap D_0 = \ker\theta_1 \cap D_0$, and $\varrho' \in \Gamma_2$ satisfying $D_0 \not\subseteq \ker(\varrho'+\mu)$ and $D_0 \not\subseteq \ker(\alpha'+\varrho'+\mu)$.

Fourthly, we determine the action of $\{x^\alpha\partial \mid \partial \in \ker\alpha \cap D_0,\ \alpha \in \Gamma_2\backslash\Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha\}$ on $\bigoplus_{\beta\in\Gamma_2} M_\beta$.

Fix any $\alpha' \in \Gamma_2\backslash\Gamma_1$ satisfying $D_0 \not\subseteq \ker\alpha'$. Write $\alpha' = k_0\theta_1 + \alpha_0$ for some $k_0 \in \mathbb{Z}\backslash\{0\}$ and $\alpha_0 \in \Gamma_1$. If $\ker\alpha' \cap D_0 \neq \ker\theta_1 \cap D_0$, then $\alpha' = k_0\theta_1 + \alpha_0$ implies $D_0 \not\subseteq \ker\alpha_0$ and $\ker\alpha_0 \cap D_0 \neq \ker\theta_1 \cap D_0$. Thus we have $\ker\theta_1 \cap \ker\alpha_0 \cap \ker\epsilon_1 \cap D_0 = \{0\}$ or $\ker\theta_1 \cap \ker\alpha_0 \cap \ker\epsilon_2 \cap D_0 = \{0\}$. Without loss of generality, we assume that $\ker\theta_1 \cap \ker\alpha_0 \cap \ker\epsilon_1 \cap D_0 = \{0\}$. Let

$$
G_3 = \mathbb{Z}\theta_1 + \mathbb{Z}\epsilon_1 + \mathbb{Z}\alpha_0. \tag{5.46}
$$

Then $(\bigcap_{\nu\in G_3}\ker\nu) \cap D_0 = \{0\}$ and G_3 can be viewed as a subgroup of D_0^*. Hence we get a simple generalized divergence-free Lie algebra $\mathcal{S}(G_3, D_0)$, which is also a subalgebra of $\mathcal{S}(\Gamma, D)$. Note that $\alpha' \in G_3$. Moreover, since $\mathcal{X}_3 = \{x^{\pm\sigma}\partial \mid \sigma \in \{\theta_1, \epsilon_1, \alpha_0\},\ \partial \in \ker\sigma \cap D_0\}$ generates $\mathcal{S}(G_3, D_0)$, and (5.26), (5.42) and (5.45) give the action of $\mathcal{X}_3$ on $\mathcal{S}(G_3, D_0)$-module $\bigoplus_{\beta\in\Gamma_2} M_\beta$, we can deduce

$$x^{\alpha'}\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha'+\beta} \tag{5.47}$$

for $\partial \in \ker\alpha' \cap D_0$, $\beta \in \Gamma_2$ such that $D_0 \not\subseteq \ker(\beta+\mu)$ and $D_0 \not\subseteq \ker(\beta+\mu+\alpha')$.

If $\ker\alpha' \cap D_0 = \ker\theta_1 \cap D_0$, then $\alpha' = k_0\theta_1 + \alpha_0$ implies $D_0 \not\subseteq \ker(\alpha_0 - k_0\epsilon_1)$ and $\ker(\theta_1+\epsilon_1)\cap D_0 \neq \ker(\alpha_0 - k_0\epsilon_1) \cap D_0$. Choose $\epsilon_3 \in \Gamma_1$ such that

$$\ker(\theta_1+\epsilon_1) \cap \ker(\alpha_0 - k_0\epsilon_1) \cap \ker\epsilon_3 \cap D_0 = \{0\}. \tag{5.48}$$

Let

$$G_4 = \mathbb{Z}(\theta_1+\epsilon_1) + \mathbb{Z}(\alpha_0 - k_0\epsilon_1) + \mathbb{Z}\epsilon_3. \tag{5.49}$$

Then $(\bigcap_{\nu\in G_4}\ker\nu) \cap D_0 = \{0\}$ and G_4 can be viewed as a subgroup of D_0^*. Hence we get a simple generalized divergence-free Lie algebra $\mathcal{S}(G_4, D_0)$, which is also a subalgebra of $\mathcal{S}(\Gamma, D)$. Note that $\alpha' \in G_4$. Moreover, since the set

$$\mathcal{X}_4 = \left\{x^{\pm\sigma}\partial \;\middle|\; \sigma \in \{\theta_1+\epsilon_1, \alpha_0 - k_0\epsilon_1, \epsilon_3\},\ \partial \in \ker\sigma \cap D_0\right\} \tag{5.50}$$

generates $\mathcal{S}(G_4, D_0)$, and (5.26), (5.42) and (5.45) give the action of $\mathcal{X}_4$ on $\mathcal{S}(G_4, D_0)$-module $\bigoplus_{\beta\in\Gamma_2} M_\beta$, we can deduce

$$x^{\alpha'}\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha'+\beta} \tag{5.51}$$

for $\partial \in \ker\alpha' \cap D_0$, $\beta \in \Gamma_2$ such that $D_0 \not\subseteq \ker(\beta+\mu)$ and $D_0 \not\subseteq \ker(\beta+\mu+\alpha')$.

To sum up, (5.42), (5.45), (5.47) and (5.51) show that Γ_2 satisfies $(p2)$ in (II). This contradicts the maximality of Γ_1. On the other hand, if Γ_1 satisfies $(p1)$ in (II), we can similarly prove that Γ_2 satisfies $(p1)$ in (II), which also contradicts the maximality of Γ_1; we omit the details here. So we must have $\Gamma_1 = \Gamma$. Therefore the lemma follows. □

Lemma 5.2. *$\mathcal{S}(\Gamma, D)$ acts on M in one of the following two ways*:

$(P'1)$ *$x^\alpha\partial.M = \{0\}$ for $\partial \in \ker\alpha$ and $\alpha \in \Gamma\backslash\{0\}$;*
$(P'2)$ *there exist $\mu \in D^*$ and $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma\backslash\{-\mu\}\}$ such that*

$$x^\alpha\partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.52}$$

for $\partial \in \ker\alpha$, $\alpha \in \Gamma\backslash\{0\}$ and $\beta \in \Gamma$ satisfying $\beta+\mu \neq 0 \neq \beta+\mu+\alpha$.

Proof. Lemma 5.1 and Zorn's lemma imply that, there exists a subspace $D' \subseteq D$ which is maximal with the property:

(I′) $\dim D' \geqslant 3$,
(II′) $\{x^\alpha\partial \mid \partial \in \ker\alpha \cap D',\ \alpha \in \Gamma$ satisfying $D' \not\subseteq \ker\alpha\}$ act on M in one of the following two ways:

$(p'1)$ $x^\alpha \partial.M = \{0\}$ for $\partial \in \ker\alpha \cap D'$ and $\alpha \in \Gamma$ satisfying $D' \nsubseteq \ker\alpha$;
$(p'2)$ there exist $\mu \in D^*$ and $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma$ satisfying $D' \nsubseteq \ker(\beta+\mu)\}$ such that

$$x^\alpha \partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.53}$$

for $\partial \in \ker\alpha \cap D'$, $\alpha \in \Gamma$ satisfying $D' \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $D' \nsubseteq \ker(\beta+\mu)$ and $D' \nsubseteq \ker(\beta+\mu+\alpha)$.

To prove the lemma, it suffices to show $D' = D$. Suppose $D' \neq D$. We will see that this leads to a contradiction.

Pick $\bar{\partial} \in D \backslash D'$. Set

$$D'' = D' + \mathbb{F}\bar{\partial}. \tag{5.54}$$

Observe that D'' satisfies (I$'$). We will also show D'' satisfies (II$'$).

Assume D' satisfies $(p'2)$ in (II$'$); the case in which D' satisfies $(p'1)$ can be proved similarly. Namely, we can choose $\mu \in D^*$ and $\{0 \neq v_\beta \in M_\beta \mid \beta \in \Gamma$ satisfying $D' \nsubseteq \ker(\beta+\mu)\}$ such that

$$x^\alpha \partial.v_\beta = (\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.55}$$

for $\partial \in \ker\alpha \cap D'$, $\alpha \in \Gamma$ satisfying $D' \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $D' \nsubseteq \ker(\beta+\mu)$ and $D' \nsubseteq \ker(\beta+\mu+\alpha)$. Then we proceed our proof in several steps.

Firstly, we give a claim:

Pick any two linearly independent elements $\bar{\partial}_1, \bar{\partial}_2 \in D'$. Set

$$\tilde{D}_0 = \mathbb{F}\bar{\partial} + \mathbb{F}\bar{\partial}_1 + \mathbb{F}\bar{\partial}_2. \tag{5.56}$$

Lemma 5.1 and (5.55) imply that there exist $\mu' \in D^*$ and $\{0 \neq u_\beta \in M_\beta \mid \beta \in \Gamma$ satisfying $\tilde{D}_0 \nsubseteq \ker(\beta+\mu')\}$ such that

$$x^\alpha \partial.u_\beta = \big(\beta+\mu'\big)(\partial)u_{\alpha+\beta} \tag{5.57}$$

for $\partial \in \ker\alpha \cap \tilde{D}_0$, $\alpha \in \Gamma$ satisfying $\tilde{D}_0 \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $\tilde{D}_0 \nsubseteq \ker(\beta+\mu')$ and $\tilde{D}_0 \nsubseteq \ker(\beta+\mu'+\alpha)$. Similar arguments as those from (4.38) to (4.43), combined with (5.55) and (5.57), indicate

$$\mu'(\partial) = \mu(\partial) \quad \text{for } \partial \in \mathbb{F}\bar{\partial}_1 + \mathbb{F}\bar{\partial}_2. \tag{5.58}$$

Assume

$$u_\beta = \tilde{a}_\beta v_\beta \quad \text{for } \beta \in \Gamma \text{ satisfying } \mathbb{F}\bar{\partial}_1 + \mathbb{F}\bar{\partial}_2 \nsubseteq \ker(\beta+\mu), \tag{5.59}$$

where $\tilde{a}_\beta \in \mathbb{F}$.

Claim 1. *$\tilde{a}_\beta = \tilde{a}$ is independent of β. Moreover,*

$$x^\alpha \partial.v_\beta = \big(\beta+\mu'\big)(\partial)v_{\alpha+\beta} \tag{5.60}$$

for $\partial \in \ker\alpha \cap \tilde{D}_0$, $\alpha \in \Gamma$ satisfying $\tilde{D}_0 \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $\mathbb{F}\bar{\partial}_1 + \mathbb{F}\bar{\partial}_2 \nsubseteq \ker(\beta+\mu)$ and $\mathbb{F}\bar{\partial}_1 + \mathbb{F}\bar{\partial}_2 \nsubseteq \ker(\beta+\mu+\alpha)$.

Choose $\rho_0 \in \Gamma$ such that $(\rho_0 + \mu)(\bar\partial_1) \neq 0$. Write

$$\partial' = (\rho_0 + \mu)(\bar\partial_1)\bar\partial_2 - (\rho_0 + \mu)(\bar\partial_2)\bar\partial_1. \tag{5.61}$$

Then $(\rho_0 + \mu)(\partial') = 0$. For any $\alpha \in \Gamma$ such that $\alpha(\partial') \neq 0$, making use of (5.58), we deduce from (5.55) and (5.57) that

$$x^\alpha\big(\alpha\big(\partial'\big)\bar\partial_1 - \alpha(\bar\partial_1)\partial'\big).v_{\rho_0} = \alpha\big(\partial'\big)(\rho_0 + \mu)(\bar\partial_1)v_{\alpha+\rho_0} \tag{5.62}$$

and

$$x^\alpha\big(\alpha\big(\partial'\big)\bar\partial_1 - \alpha(\bar\partial_1)\partial'\big).u_{\rho_0} = \alpha\big(\partial'\big)(\rho_0 + \mu)(\bar\partial_1)u_{\alpha+\rho_0}, \tag{5.63}$$

which implies

$$\tilde a_{\alpha+\rho_0} = \tilde a_{\rho_0}. \tag{5.64}$$

Such α's do exist and we denote one of them by α_0. For any $\alpha' \in \Gamma$ such that $\alpha'(\partial') = 0$ but $(\alpha' + \rho_0 + \mu)(\bar\partial_1) \neq 0$, we have $(\alpha' - \alpha_0)(\partial') \neq 0$. Moreover, making use of (5.58), we derive from (5.55) and (5.57) that

$$x^{\alpha'-\alpha_0}\big(\alpha_0\big(\partial'\big)\bar\partial_1 + \big(\alpha' - \alpha_0\big)(\bar\partial_1)\partial'\big).v_{\rho_0+\alpha_0} = \alpha_0\big(\partial'\big)\big(\alpha' + \rho_0 + \mu\big)(\bar\partial_1)v_{\alpha'+\rho_0} \tag{5.65}$$

and

$$x^{\alpha'-\alpha_0}\big(\alpha_0\big(\partial'\big)\bar\partial_1 + \big(\alpha' - \alpha_0\big)(\bar\partial_1)\partial'\big).u_{\rho_0+\alpha_0} = \alpha_0\big(\partial'\big)\big(\alpha' + \rho_0 + \mu\big)(\bar\partial_1)u_{\alpha'+\rho_0}, \tag{5.66}$$

which implies

$$\tilde a_{\alpha'+\rho_0} = \tilde a_{\alpha_0+\rho_0} = \tilde a_{\rho_0}. \tag{5.67}$$

Since $\{\partial', \bar\partial_1\}$ forms a basis of $\mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2$, (5.64) and (5.67) show

$$\tilde a_\beta = \tilde a \quad \text{for } \beta \in \Gamma \quad \text{such that } \mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2 \not\subseteq \ker(\beta + \mu). \tag{5.68}$$

So (5.57) and (5.59) give

$$x^\alpha \partial.v_\beta = \big(\beta + \mu'\big)(\partial)v_{\alpha+\beta} \tag{5.69}$$

for $\partial \in \ker\alpha \cap \tilde D_0$, $\alpha \in \Gamma$ satisfying $\tilde D_0 \not\subseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $\mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2 \not\subseteq \ker(\beta + \mu)$ and $\mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2 \not\subseteq \ker(\beta + \mu + \alpha)$. Thus the claim follows.

Secondly, we choose some $\mu_0 \in D^*$ such that

$$\mu_0|_{D'} = \mu|_{D'} \quad \text{and} \quad \mu_0(\bar\partial) = \mu'(\bar\partial), \tag{5.70}$$

where μ' is the same as in (5.57).

Thirdly, we give another claim:

Pick another two linearly independent elements $\bar\partial_1', \bar\partial_2' \in D'$. Set

$$\tilde{D}_0' = \mathbb{F}\bar\partial + \mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2'. \tag{5.71}$$

Lemma 5.1 and (5.55) imply that there exist $\mu'' \in D^*$ and $\{0 \neq u_\beta' \in M_\beta \mid \beta \in \Gamma$ satisfying $\tilde{D}_0' \nsubseteq \ker(\beta + \mu'')\}$ such that

$$x^\alpha \partial.u_\beta' = \big(\beta + \mu''\big)(\partial) u_{\alpha+\beta}' \tag{5.72}$$

for $\partial \in \ker\alpha \cap \tilde{D}_0'$, $\alpha \in \Gamma$ satisfying $\tilde{D}_0' \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $\tilde{D}_0' \nsubseteq \ker(\beta + \mu'')$ and $\tilde{D}_0' \nsubseteq \ker(\beta + \mu'' + \alpha)$.

Claim 2. $\mu''|_{\tilde{D}_0'} = \mu_0|_{\tilde{D}_0'}$.

Similar arguments as those from (4.38) to (4.43), combined with (5.55), (5.70) and (5.72), imply

$$\mu''(\partial) = \mu(\partial) = \mu_0(\partial) \quad \text{for } \partial \in \mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2'. \tag{5.73}$$

Moreover, Claim 1 indicates that

$$x^\alpha \partial.v_\beta = \big(\beta + \mu''\big)(\partial) v_{\alpha+\beta} \tag{5.74}$$

for $\partial \in \ker\alpha \cap \tilde{D}_0'$, $\alpha \in \Gamma$ satisfying $\tilde{D}_0' \nsubseteq \ker\alpha$, and $\beta \in \Gamma$ satisfying $\mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2' \nsubseteq \ker(\beta + \mu)$ and $\mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2' \nsubseteq \ker(\beta + \mu + \alpha)$.

Pick nonzero $\partial \in \mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2$ and $\partial' \in \mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2'$ (cf. (5.56), (5.71)). Choose $\alpha \in \Gamma$ such that $\alpha(\partial) \neq 0$ and $\alpha(\partial') \neq 0$. Then

$$\bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial)}\partial \in \ker\alpha \cap \tilde{D}_0, \qquad \bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial')}\partial' \in \ker\alpha \cap \tilde{D}_0'. \tag{5.75}$$

Moreover, we choose $\beta \in \Gamma$ such that $\mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2 \nsubseteq \ker(\beta + \mu)$, $\mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2' \nsubseteq \ker(\beta + \mu)$, $\mathbb{F}\bar\partial_1 + \mathbb{F}\bar\partial_2 \nsubseteq \ker(\beta + \mu + \alpha)$ and $\mathbb{F}\bar\partial_1' + \mathbb{F}\bar\partial_2' \nsubseteq \ker(\beta + \mu + \alpha)$. Then (5.69) and (5.74) show

$$x^\alpha\left(\bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial)}\partial\right).v_\beta = \big(\beta + \mu'\big)\left(\bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial)}\partial\right) v_{\alpha+\beta}, \tag{5.76}$$

$$x^\alpha\left(\bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial')}\partial'\right).v_\beta = \big(\beta + \mu''\big)\left(\bar\partial - \frac{\alpha(\bar\partial)}{\alpha(\partial')}\partial'\right) v_{\alpha+\beta}. \tag{5.77}$$

Making use of (5.58) and (5.73), we deduce that the difference between (5.76) and (5.77) is

$$\begin{aligned}
&\alpha(\bar\partial) x^\alpha\left(\frac{\partial'}{\alpha(\partial')} - \frac{\partial}{\alpha(\partial)}\right).v_\beta \\
&\quad = \left(\big(\mu' - \mu''\big)(\bar\partial) + \alpha(\bar\partial)\cdot(\beta + \mu)\left(\frac{\partial'}{\alpha(\partial')} - \frac{\partial}{\alpha(\partial)}\right)\right) v_{\alpha+\beta}.
\end{aligned} \tag{5.78}$$

On the other hand, (5.55) shows

$$x^{\alpha}\left(\frac{\partial'}{\alpha(\partial')}-\frac{\partial}{\alpha(\partial)}\right).v_{\beta}=(\beta+\mu)\left(\frac{\partial'}{\alpha(\partial')}-\frac{\partial}{\alpha(\partial)}\right)v_{\alpha+\beta}. \tag{5.79}$$

Inserting (5.79) into (5.78) and making use of (5.70), we get

$$\mu''(\bar{\partial})=\mu'(\bar{\partial})=\mu_0(\bar{\partial}). \tag{5.80}$$

Since $\tilde{D}_0'=\mathbb{F}\bar{\partial}+\mathbb{F}\bar{\partial}_1'+\mathbb{F}\bar{\partial}_2'$, this claim follows from (5.73) and (5.80).

Fourthly, we extend the set $\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma$ *satisfying* $D'\nsubseteq\ker(\beta+\mu)\}$ *to* $\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma$ *satisfying* $D''\nsubseteq\ker(\beta+\mu_0)\}$.

Since (5.55) gives the set

$$\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma\text{ satisfying }D'\nsubseteq\ker(\beta+\mu)\} \tag{5.81}$$

and (5.70) shows

$$\{\beta\in\Gamma\mid D'\nsubseteq\ker(\beta+\mu)\}=\{\beta\in\Gamma\mid D'\nsubseteq\ker(\beta+\mu_0)\}, \tag{5.82}$$

we take v_β for $\beta\in\Gamma$ satisfying $D'\nsubseteq\ker(\beta+\mu_0)$ as they were in (5.55).

Recall that $D''=D'+\mathbb{F}\bar{\partial}$ (cf. (5.54)). Then we only need to determine v_β for $\beta\in\Gamma$ such that $D'\subseteq\ker(\beta+\mu_0)$ and $(\beta+\mu_0)(\bar{\partial})\neq 0$.

Fix two linearly independent elements $\hat{\partial}_1,\hat{\partial}_2\in D'$. Set

$$\hat{D}=\mathbb{F}\bar{\partial}+\mathbb{F}\hat{\partial}_1+\mathbb{F}\hat{\partial}_2. \tag{5.83}$$

Then Lemma 5.1, Claim 2 and (5.55) imply that there exist $\{0\neq\hat{w}_\beta\in M_\beta\mid\beta\in\Gamma$ satisfying $\hat{D}\nsubseteq\ker(\beta+\mu_0)\}$ such that

$$x^{\alpha}\partial.\hat{w}_\beta=(\beta+\mu_0)(\partial)\hat{w}_{\alpha+\beta} \tag{5.84}$$

for $\partial\in\ker\alpha\cap\hat{D}$, $\alpha\in\Gamma$ satisfying $\hat{D}\nsubseteq\ker\alpha$, and $\beta\in\Gamma$ satisfying $\hat{D}\nsubseteq\ker(\beta+\mu_0)$ and $\hat{D}\nsubseteq\ker(\beta+\mu_0+\alpha)$. Moreover, Claim 1 shows that

$$\hat{w}_\beta=\hat{a}v_\beta\quad\text{for }\beta\in\Gamma\text{ satisfying }\mathbb{F}\hat{\partial}_1+\mathbb{F}\hat{\partial}_2\nsubseteq\ker(\beta+\mu_0), \tag{5.85}$$

where $\hat{a}\in\mathbb{F}$ is a nonzero constant. We then define

$$v_\beta=\frac{1}{\hat{a}}\hat{w}_\beta\quad\text{for }\beta\in\Gamma\text{ such that }D'\subseteq\ker(\beta+\mu_0)\text{ and }(\beta+\mu_0)(\bar{\partial})\neq 0. \tag{5.86}$$

Thus (5.86) together with (5.55) gives the set

$$\{0\neq v_\beta\in M_\beta\mid\beta\in\Gamma\text{ satisfying }D''\nsubseteq\ker(\beta+\mu_0)\}. \tag{5.87}$$

Fifthly, we derive the action of $x^\alpha\partial$ on v_β for all $\partial\in\ker\alpha\cap D''$, $\alpha\in\Gamma$ satisfying $D''\nsubseteq\ker\alpha$, and $\beta\in\Gamma$ satisfying $D''\nsubseteq\ker(\beta+\mu_0)$ and $D''\nsubseteq\ker(\beta+\mu_0+\alpha)$.

Case 1. $\alpha\in\Gamma$ satisfies $D'\subseteq\ker\alpha$ and $\alpha(\bar{\partial})\neq 0$.

Subcase 1.1. $\beta \in \Gamma$ satisfies $D' \subseteq \ker(\beta+\mu_0)$, $(\beta+\mu_0)(\bar{\partial}) \neq 0$ and $(\beta+\mu_0+\alpha)(\bar{\partial}) \neq 0$.

Pick any two linearly independent elements $\partial_1', \partial_2' \in \ker\alpha \cap D'' = D'$. Set

$$\tilde{D}_1 = \mathbb{F}\bar{\partial} + \mathbb{F}\partial_1' + \mathbb{F}\partial_2'. \tag{5.88}$$

Then Lemma 5.1, Claim 2 and (5.55) imply that there exist $\{0 \neq w_\rho' \in M_\rho \mid \rho \in \Gamma$ satisfying $\tilde{D}_1 \not\subseteq \ker(\rho+\mu_0)\}$ such that

$$x^\tau \partial . w_\rho' = (\rho+\mu_0)(\partial) w_{\tau+\rho}' \tag{5.89}$$

for $\partial \in \ker\tau \cap \tilde{D}_1$, $\tau \in \Gamma$ satisfying $\tilde{D}_1 \not\subseteq \ker\tau$, and $\rho \in \Gamma$ satisfying $\tilde{D}_1 \not\subseteq \ker(\rho+\mu_0)$ and $\tilde{D}_1 \not\subseteq \ker(\rho+\mu_0+\tau)$. Since $\alpha(\bar{\partial}) \neq 0$, $(\beta+\mu_0)(\bar{\partial}) \neq 0$ and $(\beta+\mu_0+\alpha)(\bar{\partial}) \neq 0$, from (5.89) and $D' \subseteq \ker(\beta+\mu_0)$, we see that

$$x^\alpha \partial . w_\beta' = (\beta+\mu_0)(\partial) w_{\beta+\alpha}' = 0 \quad \text{for } \partial \in \mathbb{F}\partial_1' + \mathbb{F}\partial_2', \tag{5.90}$$

which implies

$$x^\alpha \partial . v_\beta = 0 = (\beta+\mu_0)(\partial) v_{\beta+\alpha} \quad \text{for } \partial \in \mathbb{F}\partial_1' + \mathbb{F}\partial_2'. \tag{5.91}$$

Since $\partial_1', \partial_2' \in \ker\alpha \cap D''$ are any two linearly independent elements, we see that

$$x^\alpha \partial . v_\beta = 0 = (\beta+\mu_0)(\partial) v_{\beta+\alpha} \quad \text{for } \partial \in \ker\alpha \cap D''. \tag{5.92}$$

Subcase 1.2. $\beta \in \Gamma$ satisfies $D' \not\subseteq \ker(\beta+\mu_0)$.

Pick $\partial_1' \in D' \setminus \ker(\beta+\mu_0)$. Moreover, we pick any $\partial_2' \in D' \setminus \mathbb{F}\partial_1'$. Set

$$\tilde{D}_2 = \mathbb{F}\bar{\partial} + \mathbb{F}\partial_1' + \mathbb{F}\partial_2'. \tag{5.93}$$

From $(\beta+\mu_0)(\partial_1') \neq 0$, $D' \subseteq \ker\alpha$ and $\alpha(\bar{\partial}) \neq 0$, we see that $\mathbb{F}\partial_1' + \mathbb{F}\partial_2' \not\subseteq \ker(\beta+\mu_0)$, $\mathbb{F}\partial_1' + \mathbb{F}\partial_2' \not\subseteq \ker(\beta+\alpha+\mu_0)$ and $\tilde{D}_2 \not\subseteq \ker\alpha$. So Claim 1 and Claim 2 show

$$x^\alpha \partial . v_\beta = (\beta+\mu_0)(\partial) v_{\beta+\alpha} \quad \text{for } \partial \in \ker\alpha \cap \tilde{D}_2 = \mathbb{F}\partial_1' + \mathbb{F}\partial_2'. \tag{5.94}$$

Since $\partial_2' \in D' \setminus \mathbb{F}\partial_1'$ is arbitrary, we deduce

$$x^\alpha \partial . v_\beta = (\beta+\mu_0)(\partial) v_{\beta+\alpha} \quad \text{for } \partial \in D' = \ker\alpha \cap D''. \tag{5.95}$$

Case 2. $\alpha \in \Gamma$ satisfies $D' \not\subseteq \ker\alpha$.

Subcase 2.1. $\beta \in \Gamma$ satisfies $D' \subseteq \ker(\beta+\mu_0)$ and $(\beta+\mu_0)(\bar{\partial}) \neq 0$.

Similarly, in analogy with Subcase 1.1, we can prove

$$x^\alpha \partial . v_\beta = 0 = (\beta+\mu_0)(\partial) v_{\beta+\alpha} \quad \text{for } \partial \in \ker\alpha \cap D'. \tag{5.96}$$

Pick $\partial_1' \in D' \setminus \ker\alpha$. Moreover, we pick any nonzero $\partial_2' \in \ker\alpha \cap (\mathbb{F}\hat{\partial}_1 + \mathbb{F}\hat{\partial}_2)$ (cf. (5.83)). Set

$$\tilde{D}_3 = \mathbb{F}\bar{\partial} + \mathbb{F}\partial_1' + \mathbb{F}\partial_2'. \tag{5.97}$$

Then Lemma 5.1, Claim 2 and (5.55) imply that there exist $\{0 \neq w_\rho' \in M_\rho \mid \rho \in \Gamma$ satisfying $\tilde{D}_3 \nsubseteq \ker(\rho + \mu_0)\}$ such that

$$x^\tau \partial . w_\rho' = (\rho + \mu_0)(\partial) w_{\tau+\rho}' \tag{5.98}$$

for $\partial \in \ker\tau \cap \tilde{D}_3$, $\tau \in \Gamma$ satisfying $\tilde{D}_3 \nsubseteq \ker\tau$, and $\rho \in \Gamma$ satisfying $\tilde{D}_3 \nsubseteq \ker(\rho + \mu_0)$ and $\tilde{D}_3 \nsubseteq \ker(\rho + \mu_0 + \tau)$. In particular,

$$x^\alpha \left(\bar{\partial} - \frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right) . w_\beta' = (\beta + \mu_0)\left(\bar{\partial} - \frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right) w_{\beta+\alpha}'. \tag{5.99}$$

Choose $\gamma \in \Gamma$ such that $\gamma(\partial_2') \neq 0$. Then (5.98) shows

$$x^\gamma \left(\bar{\partial} - \frac{\gamma(\bar{\partial})}{\gamma(\partial_2')}\partial_2'\right) . w_{\beta-\gamma}' = (\beta + \mu_0)(\bar{\partial}) w_\beta'. \tag{5.100}$$

On the other hand, as $\bar{\partial} - \frac{\gamma(\bar{\partial})}{\gamma(\partial_2')}\partial_2' \in \hat{D}$, we deduce from (5.84) that

$$x^\gamma \left(\bar{\partial} - \frac{\gamma(\bar{\partial})}{\gamma(\partial_2')}\partial_2'\right) . \hat{w}_{\beta-\gamma} = (\beta + \mu_0)(\bar{\partial}) \hat{w}_\beta. \tag{5.101}$$

Since $(\beta - \gamma + \mu_0)(\partial_2') = -\gamma(\partial_2') \neq 0$ and $\partial_2' \in \mathbb{F}\hat{\partial}_1 + \mathbb{F}\hat{\partial}_2$, we know from (5.85) that $\hat{w}_{\beta-\gamma} = \hat{a} v_{\beta-\gamma}$. While (5.86), $D' \subseteq \ker(\beta + \mu_0)$ and $(\beta + \mu_0)(\bar{\partial}) \neq 0$ imply $\hat{w}_\beta = \hat{a} v_\beta$. Inserting these results into (5.101), we get

$$x^\gamma \left(\bar{\partial} - \frac{\gamma(\bar{\partial})}{\gamma(\partial_2')}\partial_2'\right) . v_{\beta-\gamma} = (\beta + \mu_0)(\bar{\partial}) v_\beta. \tag{5.102}$$

Since $(\alpha + \gamma)(\partial_2') = \gamma(\partial_2') \neq 0$, (5.98) shows

$$x^{\alpha+\gamma} \left(\partial_1' - \frac{(\alpha+\gamma)(\partial_1')}{(\alpha+\gamma)(\partial_2')}\partial_2'\right) . w_{\beta-\gamma}' = \alpha\big(\partial_1'\big) w_{\beta+\alpha}'. \tag{5.103}$$

As $\partial_1' - \frac{(\alpha+\gamma)(\partial_1')}{(\alpha+\gamma)(\partial_2')}\partial_2' \in D'$, we derive from (5.55) that

$$x^{\alpha+\gamma} \left(\partial_1' - \frac{(\alpha+\gamma)(\partial_1')}{(\alpha+\gamma)(\partial_2')}\partial_2'\right) . v_{\beta-\gamma} = \alpha\big(\partial_1'\big) v_{\beta+\alpha}. \tag{5.104}$$

Since $\alpha(\partial_1') \neq 0$ and $(\beta + \mu_0)(\bar{\partial}) \neq 0$, comparing (5.104) with (5.103) and comparing (5.102) with (5.100), we see that

$$w_{\beta-\gamma}' = c v_{\beta-\gamma}, \qquad w_{\beta+\alpha}' = c v_{\beta+\alpha} \quad \text{and} \quad w_\beta' = c v_\beta$$
$$\text{for some nonzero constant } c. \tag{5.105}$$

So inserting (5.105) into (5.99), we get

$$x^{\alpha}\left(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right).v_{\beta}=(\beta+\mu_0)\left(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right)v_{\beta+\alpha}. \tag{5.106}$$

Since $\ker\alpha\cap D''=\mathbb{F}(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1')+\ker\alpha\cap D'$, combining (5.106) with (5.96), we get

$$x^{\alpha}\partial.v_{\beta}=(\beta+\mu_0)(\partial)v_{\beta+\alpha}\quad\text{for }\partial\in\ker\alpha\cap D''. \tag{5.107}$$

Subcase 2.2. $\beta\in\Gamma$ satisfies $D'\nsubseteq\ker(\beta+\mu_0)$ and $D'\nsubseteq\ker(\beta+\mu_0+\alpha)$.

By (5.55) we have

$$x^{\alpha}\partial.v_{\beta}=(\beta+\mu_0)(\partial)v_{\alpha+\beta}\quad\text{for }\partial\in\ker\alpha\cap D'. \tag{5.108}$$

Choose $\partial_1'\in D'\setminus\ker\alpha$ and $\partial_2'\in D'\setminus(\ker(\beta+\mu_0)\cup\ker(\beta+\alpha+\mu_0))$ such that ∂_1',∂_2' are linearly independent. Set

$$\tilde{D}_4=\mathbb{F}\bar{\partial}+\mathbb{F}\partial_1'+\mathbb{F}\partial_2'. \tag{5.109}$$

Since $\mathbb{F}\partial_1'+\mathbb{F}\partial_2'\nsubseteq\ker(\beta+\mu_0)$ and $\mathbb{F}\partial_1'+\mathbb{F}\partial_2'\nsubseteq\ker(\beta+\alpha+\mu_0)$, Claim 1 and Claim 2 imply

$$x^{\alpha}\left(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right).v_{\beta}=(\beta+\mu_0)\left(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1'\right)v_{\alpha+\beta}. \tag{5.110}$$

Since $\ker\alpha\cap D''=\mathbb{F}(\bar{\partial}-\frac{\alpha(\bar{\partial})}{\alpha(\partial_1')}\partial_1')+\ker\alpha\cap D'$, combining (5.108) with (5.110), we get

$$x^{\alpha}\partial.v_{\beta}=(\beta+\mu_0)(\partial)v_{\alpha+\beta}\quad\text{for }\partial\in\ker\alpha\cap D''. \tag{5.111}$$

Subcase 2.3. $\beta\in\Gamma$ satisfies $D'\nsubseteq\ker(\beta+\mu_0)$, $D'\subseteq\ker(\beta+\alpha+\mu_0)$ and $(\beta+\alpha+\mu_0)(\bar{\partial})\neq 0$.

By the similar arguments as those in Subcase 1.1, we can prove

$$x^{\alpha}\partial.v_{\beta}=0=(\beta+\mu_0)(\partial)v_{\beta+\alpha}\quad\text{for }\partial\in\ker\alpha\cap D'. \tag{5.112}$$

Pick $\partial_1'\in D'\setminus\ker\alpha$. Moreover, we pick any nonzero $\partial_2'\in\ker\alpha\cap(\mathbb{F}\hat{\partial}_1+\mathbb{F}\hat{\partial}_2)$ (cf. (5.83)). Set

$$\tilde{D}_5=\mathbb{F}\bar{\partial}+\mathbb{F}\partial_1'+\mathbb{F}\partial_2'. \tag{5.113}$$

Then Lemma 5.1, Claim 2 and (5.55) imply that there exist $\{0\neq w_{\rho}'\in M_{\rho}\mid\rho\in\Gamma$ satisfying $\tilde{D}_5\nsubseteq\ker(\rho+\mu_0)\}$ such that

$$x^{\tau}\partial.w_{\rho}'=(\rho+\mu_0)(\partial)w_{\tau+\rho}' \tag{5.114}$$

for $\partial\in\ker\tau\cap\tilde{D}_5$, $\tau\in\Gamma$ satisfying $\tilde{D}_5\nsubseteq\ker\tau$, and $\rho\in\Gamma$ satisfying $\tilde{D}_5\nsubseteq\ker(\rho+\mu_0)$ and $\tilde{D}_5\nsubseteq$

$\ker(\rho+\mu_0+\tau)$. In particular, since $(\beta+\mu_0)(\partial_1')=-\alpha(\partial_1')\neq 0$ and $(\beta+\alpha+\mu_0)(\bar\partial)\neq 0$, we have

$$x^\alpha\left(\bar\partial-\frac{\alpha(\bar\partial)}{\alpha(\partial_1')}\partial_1'\right).w_\beta'=(\beta+\mu_0)\left(\bar\partial-\frac{\alpha(\bar\partial)}{\alpha(\partial_1')}\partial_1'\right)w_{\beta+\alpha}'. \tag{5.115}$$

Choose $\gamma\in\Gamma$ such that $\gamma(\partial_2')\neq 0$. Then (5.114) shows

$$x^\gamma\left(\bar\partial-\frac{\gamma(\bar\partial)}{\gamma(\partial_2')}\partial_2'\right).w_{\beta+\alpha-\gamma}'=(\beta+\alpha+\mu_0)(\bar\partial)w_{\beta+\alpha}'. \tag{5.116}$$

On the other hand, as $\bar\partial-\frac{\gamma(\bar\partial)}{\gamma(\partial_2')}\partial_2'\in\hat D$, we deduce from (5.84) that

$$x^\gamma\left(\bar\partial-\frac{\gamma(\bar\partial)}{\gamma(\partial_2')}\partial_2'\right).\hat w_{\beta+\alpha-\gamma}=(\beta+\alpha+\mu_0)(\bar\partial)\hat w_{\beta+\alpha}. \tag{5.117}$$

Since $(\beta+\alpha-\gamma+\mu_0)(\partial_2')=-\gamma(\partial_2')\neq 0$ and $\partial_2'\in\mathbb{F}\hat\partial_1+\mathbb{F}\hat\partial_2$, we have $\hat w_{\beta+\alpha-\gamma}=\hat a v_{\beta+\alpha-\gamma}$ by (5.85). While (5.86), $D'\subseteq\ker(\beta+\alpha+\mu_0)$ and $(\beta+\alpha+\mu_0)(\bar\partial)\neq 0$ imply $\hat w_{\beta+\alpha}=\hat a v_{\beta+\alpha}$. Inserting these results into (5.117), we get

$$x^\gamma\left(\bar\partial-\frac{\gamma(\bar\partial)}{\gamma(\partial_2')}\partial_2'\right).v_{\beta+\alpha-\gamma}=(\beta+\alpha+\mu_0)(\bar\partial)v_{\beta+\alpha}. \tag{5.118}$$

Since $(\gamma-\alpha)(\partial_2')=\gamma(\partial_2')\neq 0$, (5.114) shows

$$x^{\gamma-\alpha}\left(\partial_1'-\frac{(\gamma-\alpha)(\partial_1')}{(\gamma-\alpha)(\partial_2')}\partial_2'\right).w_{\beta+\alpha-\gamma}'=-\alpha\left(\partial_1'\right)w_\beta'. \tag{5.119}$$

As $\partial_1'-\frac{(\gamma-\alpha)(\partial_1')}{(\gamma-\alpha)(\partial_2')}\partial_2'\in D'$, we derive from (5.55) that

$$x^{\gamma-\alpha}\left(\partial_1'-\frac{(\gamma-\alpha)(\partial_1')}{(\gamma-\alpha)(\partial_2')}\partial_2'\right).v_{\beta+\alpha-\gamma}=-\alpha\left(\partial_1'\right)v_\beta. \tag{5.120}$$

Since $\alpha(\partial_1')\neq 0$ and $(\beta+\alpha+\mu_0)(\bar\partial)\neq 0$, comparing (5.120) with (5.119) and comparing (5.118) with (5.116), we see that

$$w_{\beta+\alpha-\gamma}'=c'v_{\beta+\alpha-\gamma},\qquad w_\beta'=c'v_\beta\quad\text{and}\quad w_{\beta+\alpha}'=c'v_{\beta+\alpha}$$
$$\text{for some nonzero constant } c'. \tag{5.121}$$

So inserting (5.121) into (5.115), we get

$$x^\alpha\left(\bar\partial-\frac{\alpha(\bar\partial)}{\alpha(\partial_1')}\partial_1'\right).v_\beta=(\beta+\mu_0)\left(\bar\partial-\frac{\alpha(\bar\partial)}{\alpha(\partial_1')}\partial_1'\right)v_{\beta+\alpha}. \tag{5.122}$$

Since $\ker\alpha\cap D''=\mathbb{F}(\bar\partial-\frac{\alpha(\bar\partial)}{\alpha(\partial_1')}\partial_1')+\ker\alpha\cap D'$, combining (5.122) with (5.112), we find

$$x^\alpha\partial.v_\beta=(\beta+\mu_0)(\partial)v_{\beta+\alpha}\quad\text{for }\partial\in\ker\alpha\cap D''. \tag{5.123}$$

To sum up, the above two cases show

$$x^{\alpha}\partial.v_{\beta}=(\beta+\mu_0)(\partial)v_{\alpha+\beta} \tag{5.124}$$

for $\partial\in\ker\alpha\cap D''$, $\alpha\in\Gamma$ satisfying $D''\nsubseteq\ker\alpha$, and $\beta\in\Gamma$ satisfying $D''\nsubseteq\ker(\beta+\mu_0)$ and $D''\nsubseteq\ker(\beta+\mu_0+\alpha)$. Namely, D'' satisfies $(p'2)$ in (II′). This contradicts the maximality of D'.

The case in which D' satisfies $(p'1)$ in (II′) similarly leads to a contradiction. We omit the details here. So we must have $D'=D$, from which the lemma follows. □

Lemma 5.3. *The $\mathcal{S}(\Gamma,D)$-module M is isomorphic to*

(i) *$\mathscr{M}_{\mu}$ for some $\mu\in D^*$;* (5.125)

(ii) *$\mathscr{A}_{\eta}$ for some $\eta\in D^*\backslash\{0\}$;* (5.126)

(iii) *$\mathscr{B}_{\eta}$ for some $\eta\in D^*\backslash\{0\}$;* (5.127)

(iv) *$\bigoplus_{\nu\in\Gamma}\mathbb{F}w_{\nu}$, where each component is a trivial submodule of $\mathcal{S}(\Gamma,D)$.* (5.128)

Proof. If $\mathcal{S}(\Gamma,D)$ acts on M as in $(P'1)$ of Lemma 5.2, i.e.,

$$x^{\alpha}\partial.M=\{0\}\quad\text{for }\partial\in\ker\alpha\text{ and }\alpha\in\Gamma\backslash\{0\}, \tag{5.129}$$

we see that

$$M\simeq\bigoplus_{\nu\in\Gamma}\mathbb{F}w_{\nu},\quad\text{where each component is a trivial submodule of }\mathcal{S}(\Gamma,D). \tag{5.130}$$

Assume that $\mathcal{S}(\Gamma,D)$ acts on M as in $(P'2)$ of Lemma 5.2; i.e., there exist $\mu\in D^*$ and $\{0\neq v_{\beta}\in M_{\beta}\mid\beta\in\Gamma\backslash\{-\mu\}\}$ such that

$$x^{\alpha}\partial.v_{\beta}=(\beta+\mu)(\partial)v_{\alpha+\beta} \tag{5.131}$$

for $\partial\in\ker\alpha$, $\alpha\in\Gamma\backslash\{0\}$ and $\beta\in\Gamma$ satisfying $\beta+\mu\neq0\neq\beta+\mu+\alpha$.

If $\mu\in D^*\backslash\Gamma$, (5.131) implies $M\simeq\mathscr{M}_{\mu}$. Assume $\mu\in\Gamma$. Pick some $0\neq v_{-\mu}\in M_{-\mu}$. Write

$$x^{\alpha}\partial.v_{-\mu}=g(\alpha,\partial)v_{\alpha-\mu}\quad\text{and}\quad x^{\alpha}\partial.v_{-\mu-\alpha}=h(\alpha,\partial)v_{-\mu} \tag{5.132}$$

for $\alpha\in\Gamma\backslash\{0\}$ and $\partial\in\ker\alpha$. To complete the proof, we need to compute $g(\alpha,\partial)$ and $h(\alpha,\partial)$ for all $\alpha\in\Gamma\backslash\{0\}$ and $\partial\in\ker\alpha$. We now proceed our proof in three cases.

Case 1. $g(\alpha,\partial)=0$ and $h(\alpha,\partial)=0$ for all $\alpha\in\Gamma\backslash\{0\}$ and $\partial\in\ker\alpha$.

By (5.131) and (5.132), we have $M\simeq\mathscr{M}_{\mu}$.

Case 2. $g(\alpha_0,\partial_0)\neq0$ for some $\alpha_0\in\Gamma\backslash\{0\}$ and $\partial_0\in\ker\alpha_0$.

Applying (5.131) and (5.132) to

$$\big[x^{\alpha_0}\partial_0, x^{\beta}\partial\big].v_{-\mu-\beta} = x^{\alpha_0+\beta}\big(\beta(\partial_0)\partial - \alpha_0(\partial)\partial_0\big).v_{-\mu-\beta}, \tag{5.133}$$

we get

$$g(\alpha_0, \partial_0)h(\beta, \partial) = 0 \quad \text{for } \beta \in \Gamma \backslash \{0, \alpha_0\},\ \partial \in \ker\beta, \tag{5.134}$$

which implies

$$h(\beta, \partial) = 0 \quad \text{for } \beta \in \Gamma \backslash \{0, \alpha_0\},\ \partial \in \ker\beta. \tag{5.135}$$

Fix any $\gamma_1, \gamma_2 \in \Gamma\backslash\{0\}$ such that $\ker\gamma_1 \neq \ker\gamma_2$. Pick any nonzero $\tilde{\partial} \in \ker\gamma_1 \cap \ker\gamma_2$, $\partial_1 \in \ker\gamma_1 \backslash \ker\gamma_2$ and $\partial_2 \in \ker\gamma_2 \backslash \ker\gamma_1$. Inserting (5.132) into

$$\big[x^{-\gamma_1}\partial_1, \big[x^{\gamma_2}\partial_2, x^{\gamma_1}\tilde{\partial}\big]\big].v_{-\mu} = \gamma_2(\partial_1)\gamma_1(\partial_2)x^{\gamma_2}\tilde{\partial}.v_{-\mu}, \tag{5.136}$$

we obtain

$$\gamma_2(\partial_1)\gamma_1(\partial_2)g(\gamma_1, \tilde{\partial}) - g(-\gamma_1, \partial_1)h(\gamma_1, \tilde{\partial})g(\gamma_2, \partial_2) = \gamma_2(\partial_1)\gamma_1(\partial_2)g(\gamma_2, \tilde{\partial}). \tag{5.137}$$

When $\gamma_1 \neq \alpha_0$, using (5.135) in (5.137), we get

$$g(\gamma_2, \tilde{\partial}) = g(\gamma_1, \tilde{\partial}). \tag{5.138}$$

When $\gamma_1 = \alpha_0$, applying (5.132) and (5.135) to $x^{\alpha_0}\tilde{\partial}.x^{-\alpha_0}\partial_1.v_{-\mu} = x^{-\alpha_0}\partial_1.x^{\alpha_0}\tilde{\partial}.v_{-\mu}$, we first have

$$g(-\alpha_0, \partial_1)h(\alpha_0, \tilde{\partial}) = g(\alpha_0, \tilde{\partial})h(-\alpha_0, \partial_1) = 0. \tag{5.139}$$

Then inserting (5.139) into (5.137), we get

$$g(\gamma_2, \tilde{\partial}) = g(\alpha_0, \tilde{\partial}). \tag{5.140}$$

Since $\tilde{\partial} \in \ker\gamma_1 \cap \ker\gamma_2$ is any nonzero element, from (5.138) and (5.140), we see that

$$g(\gamma_2, \partial) = g(\gamma_1, \partial) \tag{5.141}$$

for $\partial \in \ker\gamma_1 \cap \ker\gamma_2$ and any $\gamma_1, \gamma_2 \in \Gamma\backslash\{0\}$ satisfying $\ker\gamma_1 \neq \ker\gamma_2$.

Choose $\alpha_1 \in \Gamma\backslash\{0\}$ such that $\ker\alpha_1 \neq \ker\alpha_0$. Pick $\partial' \in \ker\alpha_1 \backslash \ker\alpha_0$. Since $\Gamma = \ker\alpha_0 + \mathbb{F}\partial'$, we define $\eta \in D^*$ by

$$\eta(\partial) = g(\alpha_0, \partial) \quad \text{for any } \partial \in \ker\alpha_0; \qquad \eta\big(\partial'\big) = g\big(\alpha_1, \partial'\big). \tag{5.142}$$

Then by (5.141) and (5.142), we get

$$g(\alpha_1, \partial) = \eta(\partial) \quad \text{for any } \partial \in \ker\alpha_1. \tag{5.143}$$

Choose $\alpha_2 \in \Gamma\backslash\{0\}$ such that $\operatorname{codim}_D(\ker\alpha_0 \cap \ker\alpha_1 \cap \ker\alpha_2) = 3$. Then $\ker\alpha_2 \neq \ker\alpha_1$, $\ker\alpha_2 \neq$

$\ker\alpha_0$ and $\ker\alpha_2 = \ker\alpha_2 \cap \ker\alpha_0 + \ker\alpha_2 \cap \ker\alpha_1$. By (5.141), (5.142) and (5.143), we derive

$$g(\alpha_2, \partial) = \eta(\partial) \quad \text{for any } \partial \in \ker\alpha_2. \tag{5.144}$$

For any $\beta \in \Gamma\backslash\{0\}$, we have $\operatorname{codim}_D(\ker\beta \cap \ker\alpha_i \cap \ker\alpha_j) = 3$ for some $i, j \in \{0, 1, 2\}$ with $i \neq j$. So (5.141)–(5.144) indicate,

$$g(\beta, \partial) = \eta(\partial) \quad \text{for any } \beta \in \Gamma\backslash\{0\} \text{ and } \partial \in \ker\beta. \tag{5.145}$$

By (5.145) we have $g(-\alpha_0, \partial_0) = \eta(\partial_0) = g(\alpha_0, \partial_0) \neq 0$. So, we get

$$h(\beta, \partial) = 0 \quad \text{for } \beta \in \Gamma\backslash\{0, -\alpha_0\} \text{ and } \partial \in \ker\beta \tag{5.146}$$

with α_0 replaced by $-\alpha_0$ in the discussion from (5.133) to (5.135). Hence, (5.131), (5.132), (5.135), (5.145) and (5.146) imply $M \simeq \mathscr{A}_\eta$.

Case 3. $g(\alpha, \partial) = 0$ for all $\alpha \in \Gamma\backslash\{0\}$ and $\partial \in \ker\alpha$, while $h(\alpha_0, \partial_0) \neq 0$ for some $\alpha_0 \in \Gamma\backslash\{0\}$ and $\partial_0 \in \ker\alpha_0$.

Fix any $\gamma_1, \gamma_2 \in \Gamma\backslash\{0\}$ such that $\ker\gamma_1 \neq \ker\gamma_2$. Pick any nonzero $\tilde{\partial} \in \ker\gamma_1 \cap \ker\gamma_2$, $\partial_1 \in \ker\gamma_1\backslash\ker\gamma_2$ and $\partial_2 \in \ker\gamma_2\backslash\ker\gamma_1$. Applying (5.131) and (5.132) to

$$\left[x^{-\gamma_1}\partial_1, \left[x^{\gamma_2}\partial_2, x^{\gamma_1}\tilde{\partial}\right]\right].v_{-\mu-\gamma_2} = \gamma_2(\partial_1)\gamma_1(\partial_2)x^{\gamma_2}\tilde{\partial}.v_{-\mu-\gamma_2}, \tag{5.147}$$

we have

$$h(\gamma_2, \tilde{\partial}) = h(\gamma_1, \tilde{\partial}). \tag{5.148}$$

Since $\tilde{\partial} \in \ker\gamma_1 \cap \ker\gamma_2$ is any nonzero element, we get

$$h(\gamma_2, \partial) = h(\gamma_1, \partial) \tag{5.149}$$

for $\partial \in \ker\gamma_1 \cap \ker\gamma_2$ and $\gamma_1, \gamma_2 \in \Gamma\backslash\{0\}$ satisfying $\ker\gamma_1 \neq \ker\gamma_2$.

Choose $\alpha_1 \in \Gamma\backslash\{0\}$ such that $\ker\alpha_1 \neq \ker\alpha_0$. Pick $\partial' \in \ker\alpha_1\backslash\ker\alpha_0$. We define $\eta \in D^*$ by

$$\eta(\partial) = h(\alpha_0, \partial) \quad \text{for any } \partial \in \ker\alpha_0; \qquad \eta(\partial') = h(\alpha_1, \partial'). \tag{5.150}$$

By the similar arguments as those from (5.142) to (5.145), we can prove

$$h(\beta, \partial) = \eta(\partial) \quad \text{for any } \beta \in \Gamma\backslash\{0\} \text{ and } \partial \in \ker\beta. \tag{5.151}$$

Hence, from (5.131), (5.132) and (5.151) we see that $M \simeq \mathscr{B}_\eta$.

Thus, we complete the proof of the lemma. □

In summary, Lemma 5.3, together with Lemma 2.1 and Theorem 2.2, implies

Lemma 5.4. *If* $\dim D \geqslant 4$, *then Theorem* 2.3 *holds.*

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References

[1] D. Dokovic, K. Zhao, Derivations, isomorphisms, and second cohomology of generalized Witt algebras, Trans. Amer. Math. Soc. 350 (2) (1998) 643–664.
[2] D. Dokovic, K. Zhao, Generalized Cartan type W Lie algebras in characteristic 0, J. Algebra 195 (1997) 170–210.
[3] D. Dokovic, K. Zhao, Generalized Cartan type S Lie algebras in characteristic 0, J. Algebra 193 (1997) 144–179.
[4] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur Lectures, Tel Aviv, 1992, in: Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.
[5] V.G. Kac, A description of filtered Lie algebras whose associated graded Lie algebras are of Cartan types, Math. USSR-Izvestija 8 (1974) 801–835.
[6] I. Kaplansky, The Virasoro algebra, Comm. Math. Phys. 86 (1982) 49–54.
[7] I. Kaplansky, L.J. Santharoubane, Harish–Chandra modules over the Virasoro algebras, Math. Sci. Res. Inst. Publ. 4 (1987) 217–231.
[8] N. Kawamoto, Generalizations of Witt algebras over a field of characteristic zero, Hiroshima Math. J. 16 (1986) 417–426.
[9] T. Larsson, Conformal fields: A class of representations of Vect(N)[J], Internat. J. Modern Phys. A 7 (26) (1992) 6493–6508.
[10] W. Lin, S. Tan, Representations of the Lie algebra for quantum torus, J. Algebra 275 (2004) 250–274.
[11] J.M. Osborn, New simple infinite-dimensional Lie algebras of characteristic 0, J. Algebra 185 (1996) 820–835.
[12] D.S. Passman, Simple Lie algebras of Witt type, J. Algebra 206 (1998) 682–692.
[13] Jeffrey Bergen, D.S. Passman, Simple Lie algebras of Special type, J. Algebra 227 (1) (2000) 45–67.
[14] S.E. Rao, Representations of Witt algebras, Publ. Res. Inst. Math. Sci. 29 (1994) 191–201.
[15] S.E. Rao, Irreducible representations of the Lie algebra of the diffeomorphisms of a d-dimensional torus, J. Algebra 182 (1996) 401–421.
[16] G. Shen, Graded modules of graded Lie algebras of Cartan type (I) – mixed product of modules, Sci. China Ser. A 29 (1986) 570–581.
[17] G. Shen, Graded modules of graded Lie algebras of Cartan type (II) – positive and negative graded modules, Sci. China Ser. A 29 (1986) 1009–1019.
[18] G. Shen, Graded modules of graded Lie algebras of Cartan type (III) – irreducible modules, Chinese Ann. Math. Ser. B 9 (1988) 404–417.
[19] Y. Su, Harish–Chandra modules of the intermediate series over the high rank Virasoro algebras and high rank super-Virasoro algebras, J. Math. Phys. 35 (1994) 2013–2023.
[20] Y. Su, Classification of Harish–Chandra modules over the super-Virasoro algebras, Comm. Algebra 23 (1995) 3653–3675.
[21] Y. Su, X. Xu, Structure of divergence-free Lie algebras, J. Algebra 243 (2001) 557–595.
[22] Y. Su, J. Zhou, Some representations of nongraded Lie algebras of generalized Witt type, J. Algebra 246 (2001) 721–738.
[23] Y. Su, K. Zhao, Generalized Virasoro and super-Virasoro algebras and modules of intermediate series, J. Algebra 252 (2002) 1–19.
[24] X. Xu, Generalizations of the Block algebras, Manuscripta Math. 100 (1999) 489–518.
[25] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, J. Algebra 224 (2000) 23–58.
[26] X. Xu, Quadratic conformal superalgebras, J. Algebra 231 (2000) 1–38.
[27] K. Zhao, Generalized Cartan type S Lie algebras in characteristic zero, II, Pacific J. Math. 192 (2000) 431–454.
[28] K. Zhao, Weight modules over generalized Witt algebras with 1-dimensional weight spaces, Forum Math. 16 (2004) 725–748, preprint, 2000.
[29] Y. Zhao, Irreducible representations of nongraded Witt type Lie algebras, J. Algebra 298 (2006) 540–562.
[30] Y. Zhao, Representations of nongraded Lie algebras of Block type, Manuscripta Math. 119 (2006) 183–216.
[31] Y. Zhao, Composition series for a family of modules of nongraded Hamiltonian type Lie algebras, J. Lie Theory 19 (2009) 1–27.
[32] Y. Zhao, Z. Liang, Representations of four-derivation Lie algebras of Block type, Acta Math. Sin. 26 (2010) 49–76.